Lecture 18. October 5, 2016

Topics: Reference frame. First law of Newton. Inertial motion and inertial frame. Noninertial reference frame. Geocentric reference frame as an example of noninertial frame. Fundamental forces and apparent forces. Body forces and surface forces. Second law of Newton. One-dimensional equation of motion. Pressure gradient force. Gravitational force. Viscous shear (shearing) and normal stresses. Viscous force. Kinematic viscosity.

Reading: Sections 1.2 and 1.3 of Holton and Hakim. Sections 5 and 6 of Fiedler.

1. Newton's first law; inertial and noninertial frames

Newton's first law of motion states that a mass (a body, an object) in uniform motion relative to coordinate system fixed in space will remain in uniform motion in the absence of any forces. Being at rest is considered to be a particular case of the uniform motion. Such uniform motion is referred to as inertial motion, and the considered fixed frame is viewed as an *inertial* (or *absolute*) reference frame. In atmospheric applications of mechanics, however, we usually use the so-called *geocentric frame* that is a reference frame at rest with respect to the moving Earth. It is clear, though, that the object at rest with respect to the moving Earth is not at rest or in uniform motion relative to coordinate system fixed in space. For instance, the motion that an observer in the geocentric frame views as an inertial motion, will be accelerated motion with respect to the coordinate system fixed in space. This makes the geocentric frame the *noninertial* frame. The so-called *apparent* forces/accelerations in the Newton's equations of motion, considered in the next sections of the course, account for reaction mechanisms that arise in the geocentric frame because of its acceleration.

2. Forces acting in the atmosphere

Among the forces causing acceleration of atmospheric parcels (particles), we will be distinguishing between *body forces* and *surface forces*. *Body forces* act on the center of mass of the fluid particle (parcel); their magnitudes are proportional to the mass of the parcel. Examples: gravitational force, Coriolis force, centrifugal force (we will consider all those in the next sections of the course). *Surface forces* act across the boundary surfaces of air parcels; their magnitudes are independent of the mass of the parcel. Examples: pressure gradient force, viscous friction force (considered in the next sections of the course).

3. Newton's second law; one-dimensional equation of motion

Newton's second law of motion states that the rate of change of momentum of an object, as measured relative to coordinates fixed in space, equal to the sum of all forces acting upon the object.

Let us first look at a motion (in terms of acceleration and rate of momentum change) of an object with mass m initiated by a single force $\mathbf{F}$ (which is a vector). For this case, the second law of Newton provides:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d(m\mathbf{v})}{dt} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a},$$

where the mass of the object is considered unchanging with time, $\mathbf{p} = m\mathbf{v}$ is the momentum of the object (it is a vector), $\mathbf{v}$ is the velocity of the object (it is also a vector), and $\mathbf{a} = \frac{d\mathbf{v}}{dt}$ is the acceleration of the object (also a vector).

In an inertial Cartesian reference frame:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{du}{dt}\mathbf{i} + \frac{dv}{dt}\mathbf{j} + \frac{dw}{dt}\mathbf{k},$$

and

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{dp_x}{dt}\mathbf{i} + \frac{dp_y}{dt}\mathbf{j} + \frac{dp_z}{dt}\mathbf{k} = \frac{d(mu)}{dt}\mathbf{i} + \frac{d(mv)}{dt}\mathbf{j} + \frac{d(mw)}{dt}\mathbf{k} = ma_x\mathbf{i} + ma_y\mathbf{j} + ma_z\mathbf{k}.$$

It is often useful to consider a case of one-dimensional motion, e.g., along the x axis. In this case:

$$F \equiv F_x = \frac{dp}{dt} = \frac{d(mu)}{dt} = ma.$$

where $a \equiv a_x$ and $p \equiv p_x$, and

$$\frac{du}{dt} = \frac{1}{m}F(x,t) = a(x,t).$$

We also know that velocity u is the rate of change of the distance x traveled by the object:

$$u = \frac{dx}{dt}.$$

Knowing the acceleration, these equations allow calculation of velocities and distances.

4. Pressure gradient force

Pressure gradient force is one of surface forces acting in the atmosphere (see p. **2**).

We consider an air volume element $\delta V = \delta x \delta y \delta z$ in the Cartesian coordinate system. Momentum imparted by molecular motions to the walls of this element per unit time per unit area (force per unit area) is the pressure. Considering pressure difference δp at the opposing walls of the element along *X*, the x component of the associated pressure gradient force $\mathbf{F}_p$ can be expressed as

$$F_{px} = -\delta p \delta y \delta z.$$

Taking into account that mass m of the considered volume is

$$m = \rho \delta x \delta y \delta z,$$

where ρ is the density of the air, and switching form finite differences to differentials, the *x* component of the pressure gradient acceleration a_{px} (that is the pressure gradient force per unit mass) may be expressed as

$$a_{px} = F_{px} / m = -\frac{1}{\rho}\frac{\partial p}{\partial x}.$$

In a similar way, the other two components of the pressure gradient force (pressure gradient acceleration) can be expressed, which results in the following expression for the total pressure gradient force per unit mass of the volume:

$$\mathbf{a}_p = \mathbf{F}_p / m = -\frac{1}{\rho}\nabla p.$$

This force per unit mass (acceleration) is thus proportional to the gradient of the pressure field, ∇p.

5. Gravitational force

Gravitational force is a body force (see p. **2**).

If two mass elements *M* and *m* are separated by a distance (usually interpreted as the distance between the centers of mass of the mass elements) $r=|\mathbf{r}|$ (with the vector $\mathbf{r}$ directed toward *m*) then the *gravitational force* $\mathbf{F}_g$ exerted by mass *M* on mass *m* is given by

$$\mathbf{F}_g = -G\frac{Mm}{r^2}\hat{\mathbf{r}} = -G\frac{Mm}{r^2}\frac{\mathbf{r}}{r},$$

where G is the so-called universal gravitational constant.

Therefore, the gravitational acceleration $\mathbf{a}_g$ experienced by the element with mass *m* (the gravitational force per unit mass) can be expressed as

$$\mathbf{a}_g = \frac{\mathbf{F}_g}{m} = -G\frac{M}{r^2}\hat{\mathbf{r}}.$$

In atmospheric applications, *M* is considered to be the mass of the Earth (regarded as a sphere with radius *a*). The distance between the center of the Earth and the object (element) with mass *m* located at the level *z* above mean sea level (this is a usual way to evaluate vertical distances in the atmosphere) is then $r = a + z$, and we come to

$$\mathbf{a}_g = -G\frac{M}{(z+a)^2}\hat{\mathbf{r}} = -G\frac{M}{a^2}\hat{\mathbf{r}}\frac{1}{(1+z/a)^2} = \frac{\mathbf{g}_0^*}{(1+z/a)^2} \equiv \mathbf{g}^*,$$

where

$$\mathbf{g}_0^* = -G\frac{M}{a^2}\hat{\mathbf{r}}$$

is the gravitational force per unit mass at mean sea level. Because throughout the atmosphere $z \ll a$, one may usually assume gravitational acceleration to be equal to its sea-level value: $\mathbf{g}^* = \mathbf{g}_0^*$.

Due to the fact that Z axis is (usually) directed along the Earth's radius: $g_x^* = g_y^* = 0$ and $g_z^* = -\left|\mathbf{g}^*\right|$.

6. Viscous shear (shearing) and normal stresses

Viscosity is a physical property of any real fluid that is associated with internal friction between the fluid particles. This friction resists the tendency of fluid to flow.

A viscous force F acting along a facet of a moving air parcel (that is, tangential to it) is associated with a *shear* (or *shearing*) *stress*

$$\tau = \frac{F}{A},$$

where A is the facet area. Thus, the stress is expressed in units of force per unit area (like pressure).

Consider a fluid that moves horizontally with velocity $u(z)$, i.e., there are no fluid motions in other coordinate directions, and u does not depend on x and y. For the case of the so-called Newtonian fluid (atmospheric air is a good example of such fluid), the resulting viscous shear stress τ_{zx} in x direction is proportional to the velocity gradient in the direction normal to the direction of the fluid motion and is given by

$$\tau_{zx} = \mu \frac{\partial u}{\partial z},$$

where the proportionality coefficient μ is called the (coefficient of) *dynamic viscosity*. It has units of force per unit area times time, $\mathrm{N\,m^{-2}\,s}$, or $\mathrm{kg\,m^{-1}\,s^{-1}}$. Subscript x in this case signifies the direction of stress and subscript z signifies the direction of velocity gradient.

General expressions for viscous shear stress components in the Newtonian fluid, for the case when velocity field has all three components changing in space, are

$$\tau_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),\ \tau_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right),$$

$$\tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),\ \tau_{yz} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right),$$

$$\tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right),\ \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right).$$

A viscous force acting normal to the facet of a moving air parcel is associated with a *normal stress*. Normal stress components in the Newtonian fluid are given by

$$\tau_{xx} = -p + 2\mu\frac{\partial u}{\partial x},\ \tau_{yy} = -p + 2\mu\frac{\partial v}{\partial y},\ \tau_{zz} = -p + 2\mu\frac{\partial w}{\partial z},$$

with

$$p = -(\tau_{xx} + \tau_{yy} + \tau_{zz})/3,$$

where p is pressure. Hence, the combination of normal stresses taken with negative sign is equal to pressure *p* (**note** in this connection the remarkable coincidence of pressure and stress units!) In the flow with no surface friction among fluid particles (where the shear stress components are all zero):

$$\tau_{xx} = \tau_{yy} = \tau_{zz} = -p\,.$$

4. Viscous force per unit mass; kinematic viscosity

Consider shear stresses at the walls of a differential volume element (like the one considered in the example of the pressure gradient force in p. **1**). If we calculate resulting viscous friction force exerted on the volume in the *x* direction and refer it to the mass $m = \rho\delta x\delta y\delta z$ of fluid in considered volume, we obtain the following expression for the resulting *viscous force* per unit mass (that is the viscous acceleration) in *x* direction:

$$F_{zx} = \frac{1}{\rho}\frac{\partial\tau_{zx}}{\partial z} = \frac{1}{\rho}\frac{\partial}{\partial z}\mu\frac{\partial u}{\partial z}\,.$$

Like in the case of the pressure-gradient force, it follows from the physical nature of the viscous force that this is a surface force.

Usually μ may be taken constant (like it is assumed in practically all atmospheric applications), and the above relationship may be rewritten as

$$F_{zx} = \frac{\mu}{\rho}\frac{\partial^2 u}{\partial z^2} \equiv \nu\frac{\partial^2 u}{\partial z^2}\,,$$

where we introduced a new quantity $\nu \equiv \dfrac{\mu}{\rho}$ called the (coefficient of) *kinematic viscosity*. This physical quantity has dimension of length squared per unit time and the SI unit of $\mathrm{m}^2\ \mathrm{s}^{-1}$.

For standard atmospheric conditions at sea level: $\nu = 1.46\cdot 10^{-5}\mathrm{m}^2\ \mathrm{s}^{-1}$.

Considering all possible combinations of mutual orientations of the viscous stress and velocity-component gradients (there will be nine combinations in total due to three orientation possibilities for each quantity), one may obtain the following expressions for the individual components of the vector $\mathbf{F}_r = F_{rx}\mathbf{i} + F_{ry}\mathbf{j} + F_{rz}\mathbf{k}$ of viscous force per unit mass (viscous acceleration):

$$F_{rx} = \frac{1}{\rho}\frac{\partial\tau_{xx}}{\partial x} + \frac{1}{\rho}\frac{\partial\tau_{yx}}{\partial y} + \frac{1}{\rho}\frac{\partial\tau_{zx}}{\partial z} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \nu\nabla^2 u\,,$$

$$F_{ry} = \frac{1}{\rho}\frac{\partial\tau_{xy}}{\partial x} + \frac{1}{\rho}\frac{\partial\tau_{yy}}{\partial y} + \frac{1}{\rho}\frac{\partial\tau_{zy}}{\partial z} = \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = \nu\nabla^2 v\,,$$

$$F_{rz} = \frac{1}{\rho}\frac{\partial\tau_{xz}}{\partial x} + \frac{1}{\rho}\frac{\partial\tau_{yz}}{\partial y} + \frac{1}{\rho}\frac{\partial\tau_{zz}}{\partial z} = \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \nu\nabla^2 w\,,$$

where u, v, w are x, y, z components of the velocity vector $\mathbf{U}$, quantities τ_{xx}, τ_{yy}, and τ_{zz} are normal stress components, and $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$, $\tau_{yz} = \tau_{zy}$ are shear stress components. The above expressions are valid under the previously adopted assumption of μ being constant plus the assumption of the constancy of density, the so-called incompressibility assumption, which also results in

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

as will be shown in the next sections of the course.

In the main portion of the atmosphere, spatial derivatives of velocity components associated with representative scales of motion are too small for the viscous stresses/force to be significant. Normally, in atmospheric considerations, this force is taken into account only within a thin layer (the so-called *viscous sublayer)* very close (in the order range from millimeters to centimeters) to the Earth's surface where the flow shears may be very strong.