Lecture 23. October 19, 2016

Topics: Structure of the static atmosphere. Equation of state. Hydrostatic equation. Relation to the geopotential change with height. Hypsometric equation. Geopotential height. Vertical coordinates related to the atmospheric pressure. Pressure gradient in the isobaric coordinate system. Generalized vertical coordinate.

Reading: Section 1.4 of Holton and Hakim.

1. Equation of state

Pressure p, absolute temperature T, and density ρ of the atmospheric air, which is considered as an ideal gas (this assumption is usually fulfilled with a high degree of accuracy), are related as

$$p = \rho RT \,,$$

where $R = 287$ J kg^{-1}s^{-1} is the so-called gas constant for dry air. This equation is called the *equation of state*. It can be written also in terms of *specific volume* $\alpha \equiv 1/\rho$ as

$$p\alpha = RT \,.$$

2. Hydrostatic equation

From the consideration of hydrostatic force balance in the column of fluid with no vertical acceleration: $-\delta p = \rho g \delta z$, where g is the gravity acceleration, see Fig. 1.9 in the textbook. In differential form, it gives the *hydrostatic equation*,

$$\partial p / \partial z = -\rho g \,,$$

which in the case of pressure being a function of the vertical coordinate only, $p=p(z)$, (which is an excellent approximation for many meteorological applications), is written as

$$dp / dz = -\rho g \,.$$

Integrating the above equation over the vertical from some level z up to the top of the atmosphere (that formally corresponds to the level $z=\infty$, where $p=0$), we obtain:

$$p(z) = \int_z^\infty \rho g dz \,.$$

Because ρg has the meaning of force per unit volume, the value of pressure at any given level in the atmosphere is equal to the weight of the unit cross section column of the air overlying this level. The average (over the globe) of this weight is taken as the *mean sea-level pressure* $p_0 = 1013.25\text{hPa}=1013.25\text{mb}$.

Note that in meteorological applications the value of sea-level pressure p_0 is often approximated by the value of $1013\ \text{hPa}$ or even $1000\ \text{hPa}$.

3. Relation to the geopotential change with height

In meteorology, the hydrostatic balance is often expressed in terms of geopotential (see Class 20) and pressure instead of pressure and geometric height *z*.

Recalling that the differential of geopotential is given by $d\Phi = gdz$ (considering the geopotential as a function of *z* only), combining the latter expression with the hydrostatic balance equation $dp/dz = -\rho g$, which results in $d\Phi/dz = -(1/\rho)dp/dz$, and performing the integration over the vertical between two levels $z_2 > z_1$ with pressure values $p(z_2) = p_2 < p_1 = p(z_1)$,

$$\int_{\Phi_1}^{\Phi_2} d\Phi = \Phi_2(z_2) - \Phi_1(z_1) = \int_{z_1}^{z_2} gdz = -\int_{p_1}^{p_2} \frac{dp}{\rho} = -\int_{p_1}^{p_2} RT\frac{dp}{p},$$

we obtain the hypsometric equation:

$$\Phi(z_2) - \Phi(z_1) = R\int_{p_2}^{p_1} Td\ln p\ .$$

Taking $z_1 = 0$ (i.e. at the sea level), $z_2 = z$, $p_1 = p_0$, and $p_2 = p$, we come to

$$\int_0^z gdz = g_0\int_0^z \frac{g}{g_0}dz = \Phi(z) = -R\int_{p_0}^{p} Td\ln p\ .$$

where g_0 is some constant value of the gravity acceleration.

4. Hypsometric equation

An important physical quantity associated with geopotential is the *geopotential height*

$$Z \equiv \frac{\Phi(z)}{g_0} = \frac{1}{g_0}\int_0^z gdz = \frac{\bar{g}}{g_0}\int_0^z dz = \frac{\bar{g}}{g_0}z,$$

where $\bar{g}$ is a mean value of *g* within the atmospheric layer from 0 to *z*. Usually, g_0 in the expression for geopotential height is taken equal to $9.81\ \text{m s}^{-2}$, which is the mean value of the gravity acceleration magnitude at mean sea level. These considerations indicate that Z and z are close to each other for elevations not very far away from the sea level. Accordingly,

$$Z_2 - Z_1 = \frac{\Phi_2}{g_0} - \frac{\Phi_1}{g_0} = \frac{R}{g_0}\int_{p_2}^{p_1} Td\ln p\ .$$

One may now define the *mean temperature* of the atmospheric layer between z_1 and z_2 in the following way

$$\langle T \rangle = \frac{\int_{p_2}^{p_1} T d \ln p}{\int_{p_2}^{p_1} d \ln p},$$

and the so-called *mean scale height*

$$H = \langle T \rangle \frac{R}{g_0}.$$

In terms of the above defined quantities:

$$Z_2 - Z_1 = \langle T \rangle \frac{R}{g_0} \ln \frac{p_1}{p_2} = H \ln \frac{p_1}{p_2}.$$

This means that the thickness of the layer bounded by isobaric surfaces is proportional to the mean temperature of the layer.

Note also that, as follows from the above equations,

$$Z = -H \ln \frac{p}{p_0}.$$

This means that in the atmosphere with T = const (where therefore $\langle T \rangle = T$ is constant and $H = TR / g_0$ is also constant) pressure decreases exponentially with height as

$$p = p_0 e^{-Z/H}.$$

5. Relations between changes of pressure in different directions

In p. **2** we considered the expression of the (vertical) hydrostatic balance,

$$\partial p / \partial z = -\rho g,$$

that provides an approximation for the vertical dependence of pressure in the real atmosphere in a form of a single-valued, monotonic relationship between p and z. Such features make it attractive from the point of view of using it for setting up a new vertical coordinate, namely the pressure itself.

Indeed, using the schematic in Fig. 1.10 of the textbook one may express the components of the horizontal pressure gradient as

$$(\partial p / \partial x)_z = -(\partial p / \partial z)_x (\partial z / \partial x)_p \text{ and } (\partial p / \partial y)_z = -(\partial p / \partial z)_y (\partial z / \partial y)_p,$$

where subscripts denote variables that are kept constant while calculation of the partial derivatives. Keep in mind also the choice of signs and the fact that $\partial p / \partial z$ is negative in the coordinate system where z is directed upward.

6. Components of pressure gradient force in the isobaric coordinate system

Recalling that

$$\partial p / \partial z = -\rho g\ ,$$

(it is the hydrostatic balance equation), and

$$\partial \Phi / \partial z = g\ ,$$

(it is the vertical component of the geopotential $\nabla\Phi = \mathbf{i}\dfrac{\partial\Phi}{\partial x} + \mathbf{j}\dfrac{\partial\Phi}{\partial y} + \mathbf{k}\dfrac{\partial\Phi}{\partial z} = -\mathbf{g}$, where $\mathbf{g} = -g\mathbf{k}$ and $g \equiv |\mathbf{g}|$, see Class 20), and using expression for $(\partial p / \partial x)_z$ from p. **5**, we obtain:

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = \frac{1}{\rho}\left(\frac{\partial p}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_p = -g\left(\frac{\partial z}{\partial x}\right)_p = -\left(\frac{\partial \Phi}{\partial x}\right)_p$$

and, by analogy,

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial y}\right)_z = -\left(\frac{\partial \Phi}{\partial y}\right)_p .$$

Two important observations:

a. Components of the horizontal pressure gradient force per unit mass (components of the pressure gradient acceleration) in the *z* coordinate system correspond to the horizontal components of the geopotential gradient in the *p* (*isobaric*) coordinate system.

b. Density does not show up in the expressions for the components of the horizontal pressure gradient force expressed through the gradients of the geopotential.

7. Generalized vertical coordinate; pressure gradient in generalized coordinate system.

The σ coordinate, which is rather popular in meteorological applications, is defined as

$$\sigma = p\ (x, y, z, t)/\ p_s\,(x, y, t),$$

where p_s is the surface pressure (**note** that this quantity is considered to be independent of *z*).

The generalized vertical coordinate *s* specified as *s*=*s*(*x*, *y*, *z*, *t*), which is supposed to be a single-valued monotonic function of height. In such a coordinate system, the components of the horizontal pressure gradient can be expressed as

$$(\partial p / \partial x)_s = (\partial p / \partial z)(\partial z / \partial x)_s + (\partial p / \partial x)_z \text{ and } (\partial p / \partial y)_s = (\partial p / \partial z)(\partial z / \partial y)_s + (\partial p / \partial y)_z\ ,$$

see Fig. 1.11 in the textbook for the geometric interpretation of the above relationships. Using the identity

$$\partial p / \partial z \equiv (\partial p / \partial s)(\partial s / \partial z),$$

we can write down the above expressions as

$$(\partial p / \partial x)_s = (\partial p / \partial x)_z + (\partial s / \partial z)(\partial z / \partial x)_s (\partial p / \partial s),$$

$$(\partial p / \partial y)_s = (\partial p / \partial y)_z + (\partial s / \partial z)(\partial z / \partial y)_s (\partial p / \partial s).$$