Lecture 24. October 24, 2016

Topics: Lagrangian and Eulerian frames. Characteristics of mass, momentum, and thermodynamic energy changes in the process of motion. Total differentiation of a scalar field. Total differentiation of vector in rotating frame.

Reading: Chapter 2 of Holton and Hakim.

1. Lagrangian and Eulerian frames

In the next sections of the course we will be considering budgets of mass, momentum, heat, and energy for a control volume of atmospheric air. We may associate this volume with the air parcel or, more generally, with a fluid parcel, see Class 2. In fluid mechanics, there are two commonly adopted conceptual set-ups (called frames) to consider motion of such a control volume (particle, parcel) and its interactions with surrounding fluid.

In the so-called *Eulerian* frame, the control volume is a parallelepiped (rectangular box) with sides $\delta x,\ \delta y,\ \delta z$, whose position is fixed relative to the coordinate system. An example of such a volume was considered in Class 19 while deriving an expression for the pressure gradient force, see Fig. 1.2 in the textbook. Transport of mass, momentum, and heat to and from the volume is maintained by the corresponding fluxes (see p. **2**) through the sides of the volume.

In the *Lagrangian* frame, the control volume is an infinitesimally small fluid parcel (particle) following the motion of the fluid. Mass, momentum, and energy of the particle are considered changing in the process of the motion.

2. Characteristics of mass, momentum, and thermodynamic energy changes in the process of motion

We consider the rates of change with time of mass, momentum, and thermodynamic energy in the process of motion of individual fluid parcels (particles).

The change of mass is usually specified in terms of changes of density ρ (it is a scalar!) that is mass per unit volume of fluid.

The change of momentum $\mathbf{p} = m\mathbf{U}$ (this quantity is a vector!) is usually considered per unit mass (m) of the fluid and is thus considered as the change of velocity $\mathbf{U} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$ (also a vector!).

Note that $\frac{d\mathbf{U}}{dt}$ has a meaning of acceleration (another vector) and $\rho\mathbf{U}$ (a vector) has (***i***) a meaning of momentum per unit volume and (***ii***) a meaning of *mass flux* (mass transported through a surface normal to $\mathbf{U}$ per unit surface area per unit time; see Class 15).

The quantities

$$\mathbf{F}_{mx} = \rho u \mathbf{U}, \; \mathbf{F}_{my} = \rho v \mathbf{U}, \; \mathbf{F}_{mz} = \rho w \mathbf{U},$$

all of which are vectors, are *fluxes of momentum* (amounts of momentum carried per unit area per unit time) in *x*, *y*, and *z* directions.

The thermodynamic (thermal) energy (a scalar) per unit mass is related through the specific heat to the temperature *T* (it is also a scalar). Indeed, $c_p T$ (called the *enthalpy*) represents amount of thermal energy per unit mass, so $mc_p T$ will have a meaning of thermal energy, while $\rho c_p T$ will have a meaning of thermal energy per unit volume.

The quantity $\rho c_p T\mathbf{U}$ thus has a meaning of thermal energy per unit area per unit time ($\mathrm{J\,m^{-2}s^{-1}}$ or $\mathrm{W\,m^{-2}}$), that is the *thermal energy* (or *heat*) *flux*.

3. Total differentiation of a scalar field

We use temperature *T* as a scalar field variable to analyze changes of scalar field in time and space associated with motion of the fluid.

Considering the value of temperature T_0 at the location x_0, y_0, z_0, at the time moment t_0, we may express the *total* temperature change δT in the particle that moved to a new location $x_0 + \delta x$, $y_0 + \delta y$, $z_0 + \delta z$ in a time increment δt as

$$\delta T = \left(\frac{\partial T}{\partial t}\right)\delta t + \left(\frac{\partial T}{\partial x}\right)\delta x + \left(\frac{\partial T}{\partial y}\right)\delta y + \left(\frac{\partial T}{\partial z}\right)\delta z + \text{h.o.t.},$$

where h.o.t. stands for "higher-order terms". After dividing by δt and tending δt to zero ($\delta t \to 0$), we come to the following expression for the total temperature change in the fluid particle following the fluid motion:

$$\frac{dT}{dt} \equiv \lim_{\delta t \to 0} \frac{\delta T}{\delta t} = \frac{\partial T}{\partial t} + \left(\frac{\partial T}{\partial x}\right)\frac{dx}{dt} + \left(\frac{\partial T}{\partial y}\right)\frac{dy}{dt} + \left(\frac{\partial T}{\partial z}\right)\frac{dz}{dt}.$$

This temperature change related to time is called the *total* (also, *substantial*, or *material*) derivative, while local temperature change expressed as $\partial T / \partial t$ is named the *local* derivative. Because the location (*x*, *y*, *z*) of the particle is dependent on time: *x*=*x*(*t*), *y*=*y*(*t*), *z*=*z*(*t*), we may introduce the components of the particle velocity as

$$u \equiv \frac{dx}{dt}, \; v \equiv \frac{dy}{dt}, \; w \equiv \frac{dz}{dt},$$

and write

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \frac{\partial T}{\partial t} + \mathbf{U} \cdot \nabla T,$$

where $\mathbf{U} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$ is the velocity vector and

$$\nabla T = \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k}$$

is the temperature gradient (a vector).

4. Total differentiation of vector in rotating frame

Let us consider an arbitrary vector $\mathbf{A} = \mathbf{i}' A_x' + \mathbf{j}' A_y' + \mathbf{k}' A_z'$ in an inertial coordinate frame with fixed in time coordinate directions. The total derivative (its expression for a scalar was derived in p. **3**) of a vector $\mathbf{A}$ in this frame with fixed coordinate axes is defined as

$$\frac{d_a\mathbf{A}}{dt} \equiv \left(\frac{d\mathbf{A}}{dt}\right)_{\text{inertial}} = \mathbf{i}'\frac{dA_x'}{dt} + \mathbf{j}'\frac{dA_y'}{dt} + \mathbf{k}'\frac{dA_z'}{dt}.$$

Let us now consider the same vector $\mathbf{A}$, but in a frame rotating with angular velocity $\mathbf{\Omega}$ with respect to the considered inertial frame. The vector will appear in the rotating frame as $\mathbf{A} = \mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z$. Its total derivative in the inertial frame, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are now changing with time, will be

$$\frac{d_a\mathbf{A}}{dt} = \frac{d_a}{dt}(\mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z) = \mathbf{i}\frac{dA_x}{dt} + \mathbf{j}\frac{dA_y}{dt} + \mathbf{k}\frac{dA_z}{dt} + \frac{d_a\mathbf{i}}{dt}A_x + \frac{d_a\mathbf{j}}{dt}A_y + \frac{d_a\mathbf{k}}{dt}A_z,$$

where the derivatives of unit coordinate vectors appear because the rotating system is in motion with respect to the inertial system. The first three terms on the right-hand side of the above expression correspond to the total derivative of vector $\mathbf{A}$ in the rotating system,

$$\frac{d\mathbf{A}}{dt} = \mathbf{i}\frac{dA_x}{dt} + \mathbf{j}\frac{dA_y}{dt} + \mathbf{k}\frac{dA_z}{dt},$$

and the rest three describe the contribution to the total derivative in the inertial system from the motion of this system, so

$$\frac{d_a\mathbf{A}}{dt} = \frac{d\mathbf{A}}{dt} + \frac{d_a\mathbf{i}}{dt}A_x + \frac{d_a\mathbf{j}}{dt}A_y + \frac{d_a\mathbf{k}}{dt}A_z.$$

It is possible to show (please do it yourself following Figs. 2.1, 2.2, and corresponding explanations in the textbook) that

$$d_a\mathbf{i}/dt = \mathbf{\Omega}\times\mathbf{i},\ d_a\mathbf{j}/dt = \mathbf{\Omega}\times\mathbf{j},\ d_a\mathbf{k}/dt = \mathbf{\Omega}\times\mathbf{k},$$

See also Class 20, where a similar analysis was carried for the velocity vector of an object whirled on a string through a circle. Therefore,

$$\frac{d_a\mathbf{i}}{dt}A_x + \frac{d_a\mathbf{j}}{dt}A_y + \frac{d_a\mathbf{k}}{dt}A_z = \mathbf{\Omega}\times A_x\mathbf{i} + \mathbf{\Omega}\times A_y\mathbf{j} + \mathbf{\Omega}\times A_z\mathbf{k} = \mathbf{\Omega}\times\mathbf{A},$$

which means that

$$\frac{d_a\mathbf{A}}{dt} = \frac{d\mathbf{A}}{dt} + \mathbf{\Omega}\times\mathbf{A}$$

or

$$\frac{d\mathbf{A}}{dt} = \frac{d_a\mathbf{A}}{dt} - \mathbf{\Omega}\times\mathbf{A}.$$