Lecture 25. October 26, 2016

Topics: Total derivative of velocity in rotating frame. Momentum balance equation in rotating frame. Velocity and acceleration in spherical coordinates. Momentum balance in spherical coordinates. Momentum balance equation in component form.

Reading: Chapter 2 of Holton and Hakim.

1. Total derivative of velocity in rotating frame

Let us first consider the balance of absolute momentum in an inertial frame (it will be denoted with the subscript *a* subscript, like it was done in Class 26). The balance can be written in the form of Newton's second law as the total rate of change of the absolute velocity (that is acceleration, or force per unit mass, or momentum rate of change per unit mass) balanced by the sum of *real* forces acting per unit mass of the fluid:

$$\frac{d_a \mathbf{U}_a}{dt} = \sum \mathbf{F}.$$

The absolute velocity of a parcel moving along the Earth's surface and viewed in the inertial frame is equal to the rate of change in time of the position $\mathbf{r}$ of the moving parcel with respect to the inertial frame, assuming that vector $\mathbf{r}$ connects the center of the Earth and the parcel and directed away from the center of the Earth:

$$\frac{d_a \mathbf{r}}{dt} = \mathbf{U}_a.$$

Based on the rules of the vector differentiation considered in Class 26, we can rewrite the above expression as

$$\mathbf{U}_a = \frac{d_a \mathbf{r}}{dt} = \frac{d\mathbf{r}}{dt} + \mathbf{\Omega} \times \mathbf{r} = \mathbf{U} + \mathbf{\Omega} \times \mathbf{r},$$

where $\mathbf{U}$ is the velocity vector in the rotating frame.

Applying the same vector differentiation procedure to the velocity vector in the inertial frame, $\mathbf{U}_a$, assuming the constancy of $\mathbf{\Omega}$, and introducing vector $\mathbf{R}$ that is a vector perpendicular to the axis of rotation, with magnitude equal to the distance to the axis of rotation, we come to:

$$\frac{d_a \mathbf{U}_a}{dt} = \frac{d\mathbf{U}_a}{dt} + \mathbf{\Omega} \times \mathbf{U}_a = \frac{d}{dt}(\mathbf{U} + \mathbf{\Omega} \times \mathbf{r}) + \mathbf{\Omega} \times (\mathbf{U} + \mathbf{\Omega} \times \mathbf{r}) = \frac{d\mathbf{U}}{dt} + \mathbf{\Omega} \times \frac{d\mathbf{r}}{dt} + \mathbf{\Omega} \times \mathbf{U} + \mathbf{\Omega} \times \mathbf{\Omega} \times \mathbf{r}$$
$$= \frac{d\mathbf{U}}{dt} + 2\mathbf{\Omega} \times \mathbf{U} - \Omega^2 \mathbf{R},$$

where $2\mathbf{\Omega} \times \mathbf{U}$ term represents the Coriolis effect (acceleration; see Class 20) and $-\Omega^2 \mathbf{R}$ term represents the centripetal force (acceleration; see Class 20).

We can now write down the total time derivative of the fluid parcel as

$$\frac{d_a \mathbf{U}_a}{dt} = \sum \mathbf{F} = \frac{d\mathbf{U}}{dt} + 2\mathbf{\Omega} \times \mathbf{U} - \Omega^2 \mathbf{R}.$$

Accordingly, in the non-inertial geocentric frame (which is the rotating Earth), the velocity change in time appears as

$$\frac{d\mathbf{U}}{dt} = \frac{d_a\mathbf{U}_a}{dt} - 2\mathbf{\Omega}\times\mathbf{U} + \Omega^2\mathbf{R}\,.$$

where $-2\mathbf{\Omega}\times\mathbf{U}$ term is the Coriolis force (acceleration) and $-\Omega^2\mathbf{R}$ term is the centrifugal force (acceleration). There both forces are apparent forces.

2. Momentum balance equation in rotating frame

By incorporating in our analysis real (i.e., acting in the inertial frame) atmospheric forces, that is the pressure gradient force, the gravitational force, and the frictional force (all discussed in previous classes), we have

$$\frac{d_a\mathbf{U}_a}{dt} = -\frac{1}{\rho}\nabla p + \mathbf{g}* + \mathbf{F}_r\,,$$

where $-\frac{1}{\rho}\nabla p$ is the pressure gradient force (acceleration), $\mathbf{g}*$ is the gravitational force (acceleration), and $\mathbf{F}_r = \nu\nabla^2\mathbf{U}_a$ is the viscous frictional force (acceleration); remember that we call operator ∇^2 the Laplace operator or the Laplacian.

However, because $\mathbf{U}_a = \mathbf{U} + \mathbf{\Omega}\times\mathbf{r}$, which indicates that difference between $\mathbf{U}_a$ and $\mathbf{U}$ is linear with respect to $\mathbf{r} = (x, y, z)$ and therefore $\nabla^2(\mathbf{\Omega}\times\mathbf{r}) = \mathbf{\Omega}\times\nabla^2\mathbf{r} = 0$, we have

$$\nabla^2\mathbf{U}_a = \nabla^2\mathbf{U}\,,$$

and the friction force may be written as

$$\mathbf{F}_r = \nu\nabla^2\mathbf{U}\,.$$

This provides the following *momentum balance equation* (also called the *equation of motion*) in the rotating frame:

$$\frac{d\mathbf{U}}{dt} = -\frac{1}{\rho}\nabla p + \mathbf{g} - 2\mathbf{\Omega}\times\mathbf{U} + \mathbf{F}_r\,,$$

where $\mathbf{F}_r$ designates the frictional force, and the gravity force (acceleration) vector $\mathbf{g}$ represents the combined effect of the gravitational and centrifugal forces (see Class 20):

$$\mathbf{g} \equiv \mathbf{g}^* + \Omega^2\mathbf{R}\,.$$

3. Velocity in spherical coordinates

The departure of the Earth shape from a sphere is negligible for most meteorological considerations. Due to this, one may expand (or express in the coordinate form) the relative velocity $\mathbf{U}$ and relative acceleration $\frac{d\mathbf{U}}{dt}$ taking the Earth surface as the coordinate surface z=0.

The corresponding coordinate system is the spherical system with λ, φ, z coordinates (longitude, latitude, height above the surface). Taking unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in this system directed eastward, northward, and upward, respectively, we represent the velocity vector in the component form as

$$\mathbf{U} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w,$$

where velocity components

$$u = \frac{dx}{dt},\ v = \frac{dy}{dt},\ w = \frac{dz}{dt}$$

refer to a local coordinate system (x, y, z), where X is aligned with λ direction, so $dx = a\cos\varphi d\lambda$, and Y is aligned with φ direction, so $dy = ad\varphi$. Therefore, u and v velocity components may be rewritten as

$$u = a\cos\varphi\frac{d\lambda}{dt},\ v = a\frac{d\varphi}{dt},$$

where r=a+z (a is the Earth radius) is the distance from the sphere's (Earth's) center to the considered location in the atmosphere. In most cases of practical interest, r may be replaced by a due to the fact that z<<a.

Note that directions of the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are not constant. These directions depend on the position on the sphere (Earth). In the process of motion, the position of an air parcel changes and this change corresponds, in general sense, to $\frac{d\mathbf{i}}{dt} \neq 0,\ \frac{d\mathbf{j}}{dt} \neq 0,\ \frac{d\mathbf{k}}{dt} \neq 0$.

4. Acceleration in spherical coordinates

Acceleration as total time derivative of velocity vector can thus be written as

$$\frac{d\mathbf{U}}{dt} = \mathbf{i}\frac{du}{dt} + \mathbf{j}\frac{dv}{dt} + \mathbf{k}\frac{dw}{dt} + u\frac{d\mathbf{i}}{dt} + v\frac{d\mathbf{j}}{dt} + w\frac{d\mathbf{k}}{dt},$$

where the derivatives of unit coordinate vectors have to be expressed through the coordinates of the moving parcel on the sphere.

From geometrical considerations, see Figs. 2.1-2.4 in the textbook, one obtains the following expressions:

$$\frac{d\mathbf{i}}{dt} = u\frac{\partial\mathbf{i}}{\partial x} = \frac{u}{a\cos\varphi}\frac{\partial\mathbf{i}}{\partial\lambda} = \frac{u\tan\varphi}{a}\mathbf{j} - \frac{u}{a}\mathbf{k},$$

$$\frac{d\mathbf{j}}{dt} = u\frac{\partial\mathbf{j}}{\partial x} + v\frac{\partial\mathbf{j}}{\partial y} = \frac{u}{a\cos\varphi}\frac{\partial\mathbf{j}}{\partial\lambda} + \frac{v}{a}\frac{\partial\mathbf{j}}{\partial\varphi} = -\frac{u\tan\varphi}{a}\mathbf{i} - \frac{v}{a}\mathbf{k},$$

$$\frac{d\mathbf{k}}{dt} = u\frac{\partial\mathbf{k}}{\partial x} + v\frac{\partial\mathbf{k}}{\partial y} = \frac{u}{a\cos\varphi}\frac{\partial\mathbf{k}}{\partial\lambda} + \frac{v}{a}\frac{\partial\mathbf{k}}{\partial\varphi} = \frac{u}{a}\mathbf{i} + \frac{v}{a}\mathbf{j}.$$

Memorizing the above relationships is not required, but you should understand basic principles of their derivation. Substituting these formulas for rates of change in time of unit vectors in the equation for $\frac{d\mathbf{U}}{dt}$ and collecting terms corresponding to different coordinate directions, we come up with the final expression for the relative acceleration of an air parcel on a rotating sphere (Earth):

$$\frac{d\mathbf{U}}{dt}=\mathbf{i}\left(\frac{du}{dt}-\frac{uv\tan\varphi}{a}+\frac{uw}{a}\right)+\mathbf{j}\left(\frac{dv}{dt}+\frac{u^2\tan\varphi}{a}+\frac{vw}{a}\right)+\mathbf{k}\left(\frac{dw}{dt}-\frac{u^2+v^2}{a}\right).$$

The terms proportional to $1/a$ on the right-hand side of the above expression are called the *curvature* terms, see also Class 21, where these terms were already considered.

5. Momentum balance equation in spherical coordinates

Consider vector form of the momentum balance equation in the rotating frame (which is the rotating Earth), see p. **2**:

$$\frac{d\mathbf{U}}{dt}=-\frac{1}{\rho}\nabla p+\mathbf{g}-2\mathbf{\Omega}\times\mathbf{U}+\mathbf{F}_r\,.$$

We will now express individual terms of this equation in coordinate (component) form. The local coordinate system (x, y, z) introduced in p. **3** will be used, where X is aligned with λ (longitude), Y is aligned with φ (latitude), and z is the local vertical with z=0 fixed at the Earth's surface (mean sea level).

Rate of total velocity change in time (acceleration)

Total derivative $\dfrac{d\mathbf{U}}{dt}$ stands for the total time derivative of the relative (with respect to the surface of the Earth) velocity vector (that is the relative acceleration). We have seen in p. **4** that this derivative on the spherical Earth can be written as

$$\frac{d\mathbf{U}}{dt}=\mathbf{i}\left(\frac{du}{dt}-\frac{uv\tan\varphi}{a}+\frac{uw}{a}\right)+\mathbf{j}\left(\frac{dv}{dt}+\frac{u^2\tan\varphi}{a}+\frac{vw}{a}\right)+\mathbf{k}\left(\frac{dw}{dt}-\frac{u^2+v^2}{a}\right).$$

Therefore, x, y, and z components of the rate of total velocity change in time (acceleration) are

$$\left(\frac{d\mathbf{U}}{dt}\right)_x=\frac{du}{dt}-\frac{uv\tan\varphi}{a}+\frac{uw}{a}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-\frac{uv\tan\varphi}{a}+\frac{uw}{a},$$

$$\left(\frac{d\mathbf{U}}{dt}\right)_y=\frac{dv}{dt}+\frac{u^2\tan\varphi}{a}+\frac{vw}{a}=\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+\frac{u^2\tan\varphi}{a}+\frac{vw}{a},$$

$$\left(\frac{d\mathbf{U}}{dt}\right)_z=\frac{dw}{dt}-\frac{u^2+v^2}{a}=\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}-\frac{u^2+v^2}{a}.$$

respectively.

Pressure gradient force per unit mass (pressure gradient acceleration)

In the local coordinate system (x, y, z), the pressure gradient term is given by

$$-\frac{1}{\rho}\nabla p=-\frac{1}{\rho}\left(\mathbf{i}\frac{\partial p}{\partial x}+\mathbf{j}\frac{\partial p}{\partial y}+\mathbf{k}\frac{\partial p}{\partial z}\right).$$

Consequently, the components of the pressure gradient force (acceleration) in this system are:

$$\left(-\frac{1}{\rho}\nabla p\right)_x=-\frac{1}{\rho}\frac{\partial p}{\partial x},\ \left(-\frac{1}{\rho}\nabla p\right)_y=-\frac{1}{\rho}\frac{\partial p}{\partial y},\ \left(-\frac{1}{\rho}\nabla p\right)_z=-\frac{1}{\rho}\frac{\partial p}{\partial z}.$$

Gravity force (acceleration)

The gravity force (acceleration) vector $\mathbf{g}=\mathbf{i}g_x+\mathbf{j}g_y+\mathbf{k}g_z$ is directed along *Z* (therefore its *x* and *y* components are both equal zero: $g_x=g_y=0$) but toward smaller *z* values. This means that

$$g\equiv|\mathbf{g}|=-g_z \text{ and } \mathbf{g}=-g\mathbf{k}.$$

To summarize: $g_x=0,\ g_y=0,\ g_z=-g\simeq-9.8\ \text{m s}^{-2}$.

Coriolis force (acceleration)

In the coordinate system (*x*, *y*, *z*), the vector product $\mathbf{\Omega}\times\mathbf{U}$ is presented by

$$\left(\Omega_y w-\Omega_z v\right)\mathbf{i}+\left(\Omega_z u-\Omega_x w\right)\mathbf{j}+\left(\Omega_x v-\Omega_y u\right)\mathbf{k}.$$

Because the vector $\mathbf{\Omega}$ is directed northward along the axis of the Earth's rotation:

$$\Omega_x=0,\ \Omega_y=\Omega\cos\varphi,\ \Omega_z=\Omega\sin\varphi,$$

where $\Omega=|\mathbf{\Omega}|$. Therefore, for the components of the Coriolis force per unit mass (Coriolis acceleration) we have:

$$\left(-2\mathbf{\Omega}\times\mathbf{U}\right)_x=2\Omega v\sin\varphi-2\Omega w\cos\varphi,\ \left(-2\mathbf{\Omega}\times\mathbf{U}\right)_y=-2\Omega u\sin\varphi,\ \left(-2\mathbf{\Omega}\times\mathbf{U}\right)_z=2\Omega u\cos\varphi.$$

Friction force (acceleration)

$$\mathbf{F}_r=\mathbf{i}F_{rx}+\mathbf{j}F_{ry}+\mathbf{k}F_{rz},$$

where F_{rx}, F_{ry}, F_{rz} are, respectively, *x*, *y*, and *z* components of $\mathbf{F}_r$.

6. Momentum balance equation in component form

Collecting all considered terms together, we may write the *momentum balance equation* (also called the *equation of motion*) in rotating frame, as a system of three balance equations for the individual components of momentum:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-\frac{uv\tan\varphi}{a}+\frac{uw}{a}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+2\Omega v\sin\varphi-2\Omega w\cos\varphi+F_{rx},$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+\frac{u^2\tan\varphi}{a}+\frac{vw}{a}=-\frac{1}{\rho}\frac{\partial p}{\partial y}-2\Omega u\sin\varphi+F_{ry},$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}-\frac{u^2+v^2}{a}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g+2\Omega u\cos\varphi+F_{rz},$$

which are commonly called the *equations of motion*.

Each equation of the above set represents the momentum balance component along an individual coordinate axis, the first one – along *X* axis, the second one – along *Y* axis, and the third one – along *Z* axis. In this particular case, we consider local coordinate system (*x*, *y*, *z*), with *X* axis aligned with λ (longitude) direction, so $dx = a\cos\varphi d\lambda$, and *Y* axis aligned with φ (latitude) direction, so $dy = ad\varphi$, where *a* is the Earth's radius and *z* is elevation above the earth surface.

Note that, according to the textbook (see the footnote on p. 40), the Coriolis-force terms proportional to $\cos\varphi$ in the above system must be neglected in order to conserve the angular momentum if the distance from the center of the earth, *r*, is approximated by the Earth radius *a*.