Lecture 27. October 31, 2016

Topics: Scale analysis of the equations of horizontal motion. Geostrophic approximation and geostrophic wind. Scaling the third equation of motion (equation of vertical motion). Geostrophic approximation and geostrophic wind. Approximate forms of horizontal prognostic equations. Rossby number. Hydrostatic approximation. Pressure and density deviations from hydrostatic values.

Reading: Section 1.6 and Chapter 2 in Holton and Hakim.

1. Scale analysis of the equations of horizontal motion

General equations of motion (also called the momentum balance equations) derived in Class 27 describe atmospheric motions on a very large range of scales. The importance of particular scales of motion may be estimated through the analyses of magnitudes of terms in the scaled equations of motion. Motions of some scales can be unimportant for a given problem and thus can be excluded from consideration (filtered out, dropped off) by elimination of the corresponding terms in the equations of motion. The notions of scale analysis and scaling in atmospheric dynamics were briefly discussed in Class 2.

Let us consider characteristic scales of atmospheric motion related to a mid-latitude synoptic system:

$L \sim 10^3\ \text{km} = 10^6\ \text{m}$ is the length scale;

$H \sim 10\ \text{km} = 10^4\ \text{m}$ is the depth scale;

$U \sim 10\ \text{m s}^{-1}$ is the horizontal velocity scale;

$W \sim 1\ \text{cm s}^{-1} = 10^{-2}\ \text{m s}^{-1}$ is the vertical velocity scale;

$L/U \sim 10^5\ \text{s}$ is the time scale;

$\delta p/\rho \sim 10^3\ \text{m}^2\ \text{s}^{-2}$ is the (normalized) horizontal pressure fluctuation scale.

Now we can estimate the magnitude of each term in the first two equations of motion using the introduced scales. For $\varphi = 45°$ (exactly the mid-latitude): $f_0 = 2\Omega\sin\varphi = 2\Omega\cos\varphi \sim 10^{-4}\text{s}^{-1}$.

The resulting scale estimates for the terms (all in m s^{-2}) of the horizontal (in *X* and *Y* directions) equations of motion (see Class 27):

Total time derivatives of horizontal velocity components: $\dfrac{du}{dt} = \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z} \sim \dfrac{dv}{dt} \sim \dfrac{U^2}{L} \sim 10^{-4}$.

Coriolis terms: $-2\Omega u\sin\varphi \sim 2\Omega v\sin\varphi \sim f_0 U \sim 10^{-3}$.

Pressure gradient terms: $-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} \sim -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} \sim \dfrac{\delta p}{\rho L} \sim 10^{-3}$.

Curvature terms: $-\dfrac{uv\tan\varphi}{a} \sim \dfrac{u^2\tan\varphi}{a} \sim \dfrac{U^2}{a} \sim 10^{-5}$ and $\dfrac{uw}{a} \sim \dfrac{vw}{a} \sim \dfrac{UW}{a} \sim 10^{-8}$.

Friction terms: $F_{rx} \sim F_{ry} \sim \dfrac{\nu U}{H^2} \sim 10^{-12}$.

2. Scaling the third equation of motion (equation of vertical motion)

The third equation of motion (it presents the balance of forces along Z axis, see Class 27) reads:

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}-\frac{u^2+v^2}{a}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g+2\Omega u\cos\varphi+F_{rz}.$$

We may now evaluate individual magnitudes of terms of this equation for a synoptic-scale motion in mid-latitudes using a scaling approach similar to the one applied to analysis of terms in the equations of the horizontal motion (see p. **1**).

To do this, we again consider characteristic scales of such atmospheric motion:

L~10^3 km=10^6 m is the length scale;

H~10 km = 10^4 m is the depth scale;

U~10 m s^{-1} is the horizontal velocity scale;

W~1 cm s^{-1} $=10^{-2}$ m s^{-1} is the vertical velocity scale;

$P_0 \sim 10^3$ hPa $=10^5$ N m^{-2} is the scale of vertical pressure difference across the atmosphere;

$f_0 = 2\Omega\sin\varphi = 2\Omega\cos\varphi \sim 10^{-4}\text{s}^{-1}$ (taking $\varphi = 45°$),

and use these scales for the estimation of the terms of the above equation.

The following estimates of individual terms (all in m s^{-2}) may be obtained.

Total time derivative of w: $\dfrac{dw}{dt}=\dfrac{\partial w}{\partial t}+u\dfrac{\partial w}{\partial x}+v\dfrac{\partial w}{\partial y}+w\dfrac{\partial w}{\partial z}\sim\dfrac{UW}{L}\sim 10^{-7}$.

Coriolis term: $2\Omega u\cos\varphi \sim f_0 U \sim 10^{-3}$.

Pressure gradient term: $-\dfrac{1}{\rho}\dfrac{\partial p}{\partial z}\sim\dfrac{1}{\rho_H}\dfrac{P_0}{H}\sim 10$ (ρ_H is the density scale in the layer of depth H).

Curvature term: $-\dfrac{u^2+v^2}{a}\sim\dfrac{U^2}{a}\sim 10^{-5}$.

Gravity term: $-g \sim 10$.

Friction term: $F_{rz} \sim \dfrac{\nu W}{H^2} \sim 10^{-15}$.

3. Geostrophic approximation and geostrophic wind

Scaling considerations presented in p. **1** regarding the first two equations of motion

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-\frac{uv\tan\varphi}{a}+\frac{uw}{a}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+2\Omega v\sin\varphi+F_{rx},$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+\frac{u^2\tan\varphi}{a}+\frac{vw}{a}=-\frac{1}{\rho}\frac{\partial p}{\partial y}-2\Omega u\sin\varphi+F_{ry},$$

demonstrated that the main terms in these equations – when applied to describe the synoptic-scale motion in mid-latitudes – are the pressure gradient terms,

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} \text{ and } -\frac{1}{\rho}\frac{\partial p}{\partial y},$$

and the Coriolis terms,

$$2\Omega v\sin\varphi \text{ and } -2\Omega u\sin\varphi .$$

If only these major terms are kept in the equations of the horizontal motion, they reduce to

$$-fv=-\frac{1}{\rho}\frac{\partial p}{\partial x},\ \ fu=-\frac{1}{\rho}\frac{\partial p}{\partial y},$$

where $f = 2\Omega\sin\varphi$ is the already familiar *Coriolis parameter*.

The above approximate form of the equations of horizontal motion corresponds to the so-called *geostrophic approximation*. The atmospheric horizontal motion (wind) under this assumption is called the *geostrophic wind*, whose vector $\mathbf{V}_g$ (**note** that it has only horizontal components!) is given by

$$\mathbf{V}_g=\mathbf{i}u_g+\mathbf{j}v_g,$$

where

$$u_g=-\frac{1}{\rho f}\frac{\partial p}{\partial y} \text{ and } v_g=\frac{1}{\rho f}\frac{\partial p}{\partial x}$$

are the *x* and *y* components of the geostrophic wind.

Using properties of the vector product, the geostrophic wind vector can be written in the form:

$$\mathbf{V}_g=\mathbf{k}\times\frac{1}{\rho f}\nabla p,$$

where del operator is applied on the horizontal (*X*-*Y*) plane, and therefore the pressure gradient is given by $\nabla p=\frac{\partial p}{\partial x}\mathbf{i}+\frac{\partial p}{\partial x}\mathbf{j}$. Please be able to derive the above formula for $\mathbf{V}_g$ using properties of the vector cross product.

Geostrophic approximation works rather well for large-scale horizontal motions away from the equator and sufficiently high above the ground.

4. Approximate forms of horizontal prognostic equations.

Geostrophic approximation allows to rewrite the horizontal pressure gradient force components $-\frac{1}{\rho}\frac{\partial p}{\partial x}$ and $-\frac{1}{\rho}\frac{\partial p}{\partial y}$ as fv_g and $-fu_g$, respectively.

If we take the equations of the horizontal motion with retained acceleration terms (these terms are the next to the pressure gradient and Coriolis terms with respect to the magnitude, see the scale analysis in p. **1**), we may write them down as the following *prognostic* equations for the horizontal wind components:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = f(v - v_g),$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -f(u - u_g).$$

Thus, the horizontal acceleration on the synoptic scales of motion is proportional to the difference between the actual and geostrophic wind (the so-called *ageostrophic wind*). In vector form, these prognostic equations for horizontal motion may be written (please be able to show it yourself) as one equation:

$$\frac{d\mathbf{V}}{dt} + f\mathbf{k}\times(\mathbf{V} - \mathbf{V}_g) = \frac{d\mathbf{V}}{dt} + f\mathbf{k}\times\mathbf{V}_a = 0.$$

where $\mathbf{V} = \mathbf{i}u + \mathbf{j}v$ is the horizontal velocity vector and $\mathbf{V}_a$ is the ageostrophic wind vector with components $u - u_g$ (in x direction) and $v - v_g$ (in y direction).

5. Rossby number

The *Rossby number* Ro is introduced in atmospheric dynamics as convenient measure of the magnitude of the horizontal acceleration compared to the action of the Coriolis force. Calculating the ratio of scales introduced in p. **1** for the horizontal acceleration, $\frac{U^2}{L}$, and for the Coriolis force per unit mass, $f_0 U$, we have:

$$\mathrm{Ro} = \frac{U}{f_0 L}.$$

The small values of Ro (Ro<<1) indicate that the magnitude of acceleration is small compared to the magnitude of the Coriolis force per unit mass, and thus the geostrophic approximation is valid.

6. Hydrostatic approximation

Scale analysis of the third equation of motion (see p. **2**) indicates that to a high degree of accuracy the pressure field in the atmosphere on synoptic scales of motion is in the *hydrostatic equilibrium* that corresponds to the state when the vertical component of the pressure gradient force is balanced by the gravity force.

The pressure p_r and density ρ_r in the idealized hydrostatic atmosphere (they are also called the *standard pressure* and the *standard density*) are thus related by the hydrostatic balance equation:

$$\frac{1}{\rho_r}\frac{\partial p_r}{\partial z} = -g,$$

which provides the *hydrostatic approximation* of the third equation of motion.

7. Deviations from the hydrostatic equilibrium

It is convenient to consider actual pressure and density fields in the atmosphere, $p(x,y,z,t)$ and $\rho(x,y,z,t)$, in terms of small deviations $p'(x,y,z,t)$ and $\rho'(x,y,z,t)$ from their standard (related to each other through the hydrostatic balance and depending on z only) values $p_r(z)$ and $\rho_r(z)$, i.e., in the form:

$$p(x,y,z,t) = p_r(z) + p'(x,y,z,t),$$

$$\rho(x,y,z,t) = \rho_r(z) + \rho'(x,y,z,t).$$

In this case, taking into account that $\left|\frac{\rho'}{\rho_r}\right| << 1$ and $\frac{1}{\rho_r}\frac{\partial p_r}{\partial z} = -g$, the main terms of the third equation of motion may be written as

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} - g = -\frac{1}{\rho}\left(\frac{\partial p}{\partial z} + \rho g\right) = -\frac{1}{\rho}\left(\frac{\partial p_r}{\partial z} + \rho_r g + \frac{\partial p'}{\partial z} + \rho' g\right) = -\frac{1}{\rho}\frac{\partial p'}{\partial z} - \frac{\rho'}{\rho} g \approx -\frac{1}{\rho_r}\frac{\partial p'}{\partial z} - \frac{\rho'}{\rho_r} g .$$

For synoptic-scale motions, taking

$$-\frac{1}{\rho}\frac{\partial p'}{\partial z} \sim \frac{\delta p}{\rho_r H} \sim \frac{10^3}{1 \cdot 10^4} = 10^{-1}\ \mathrm{m\ s^{-2}} \ \text{and}\ \frac{\rho'}{\rho} g \sim \frac{10^{-2}}{1} 10 = 10^{-1}\ \mathrm{m\ s^{-2}},$$

so

$$-\frac{1}{\rho}\frac{\partial p'}{\partial z} \approx \frac{\rho'}{\rho} g,$$

and

$$\frac{\partial p'}{\partial z} \approx -\rho' g$$

appears to be a reasonable approximation for the relation between pressure and density perturbation fields in this case.