Lecture 3. August 26, 2016

Topics: Example of manipulations with units in a mathematical expression. Dimensional homogeneity. Dimension and units of the gas constant.

Reading: Sections 1.1 to 1.3 of Holton and Hakim. Sections 1 and 2 of Fiedler.

1. Example of manipulations with units in a mathematical expression

Consider the following mathematical expression for the fall speed *w* of a spherical raindrop of radius *R* in the air with density ρ_a:

$$w = \alpha \left(\frac{\rho_0}{\rho_a} R \right)^{1/2},$$

where ρ_0 is the reference air density at the sea level and α is a proportionality coefficient (we may call it the fall-speed coefficient).

If *R* is given in units of length [L] (square brackets signifying dimension of the variable), velocity is in m s^{-1} [LT^{-1}], and densities in the usual SI units of mass per unit volume, that is kg m^{-3} [ML^{-3}], what would be the unit of the fall-speed coefficient?

In other words, which unit of α would make the fall-speed formula *dimensionally homogeneous*? Dimensional homogeneity of equation means that all terms in the equation have the same physical dimensions. In terms of the considered expression, the principle of dimensional homogeneity implies that the right-hand side of the expression, $\alpha \left(\frac{\rho_0}{\rho_a} R \right)^{1/2}$, has the same physical dimension as the left side, w, which has a dimension of velocity.

We can rewrite the velocity expression as

$$\alpha = w \left(\frac{\rho_0}{\rho_a} R \right)^{-1/2},$$

which gives

$$[\alpha] = \left[\mathrm{LT}^{-1} \left(\frac{\mathrm{ML}^{-3}}{\mathrm{ML}^{-3}} \mathrm{L} \right)^{-1/2} \right] = [\mathrm{L}^{1/2}\mathrm{T}^{-1}].$$

This means that in SI the coefficient α would have a unit of $\mathrm{m}^{1/2}\mathrm{s}^{-1}$.

One could also deduce a sign of this coefficient. Conventionally, the vertical coordinate axis *z*, along which heights are measured in the atmosphere, is directed opposite to the gravitational acceleration vector. In this way, the falling (apparently, under the action of the gravity force) raindrop is moving in the direction of smaller *z*, so its velocity w should be negative. However, the $\left(\frac{\rho_0}{\rho_a}R\right)^{1/2}$ part of the velocity expression is positive, so in order to have w negative, α should be negative.

Let us evaluate this negative coefficient based on reasonable estimates of other physical quantities in the fall-speed formula and assuming that the raindrop fall happens not too far away from the surface, so one may take $\frac{\rho_0}{\rho_a} \simeq 1$. The considered formula is valid for the raindrop radii in the range from 0.5 mm to 1.0 mm (which sometimes may be specified as 500 μm to 1000 μm). The raindrop fall speed may also vary, but it is typically within a few (negative) $\mathrm{m\ s^{-1}}$. Adopting R=900 μm and $w=$-3 $\mathrm{m\ s^{-1}}$, we come up with

$$\alpha = w\left(\frac{\rho_0}{\rho_a}R\right)^{-1/2} \approx wR^{-1/2} = -3\ \mathrm{m\ s^{-1}} \times (900\mu\mathrm{m})^{-1/2} = -3\ \mathrm{m\ s^{-1}} \times (9 \cdot 10^{-4}\mathrm{m})^{-1/2} = -100\ \mathrm{m^{1/2}s^{-1}}.$$

2. Dimension and units of the gas constant

Now consider expression of the gas law for dry air:

$$p = \rho R_d T\ ,$$

where p is atmospheric pressure in Pa = N m^{-2} = kg s^{-2} m^{-1}, [ML^{-1}T^{-2}]; T is absolute air temperature in K, [Θ]; and ρ is air density in kg m^{-3}, [ML^{-3}].

You need to identify the dimension and SI unit of the variable R_d known as the gas constant for dry air.

Rewriting the gas law as

$$R_d = \frac{p}{\rho T},$$

we come to

$$[R_d] = \left[\frac{\mathrm{ML^{-1}T^{-2}}}{\mathrm{ML^{-3}}\Theta}\right] = \left[\frac{\mathrm{L^2}}{\mathrm{T^2}\Theta}\right],$$

which in SI units would correspond to m^2 s^{-2} K^{-1}.

However, in physics and meteorology another unit of R_d is commonly used, that is J kg^{-1} K^{-1}. How to make it consistent with m^2 s^{-2} K^{-1}? For this we recall that energy [E]) is dimensionally expressed as

$$[\mathrm{E}] = [\mathrm{FL}] = \left[\mathrm{MLT^{-2}L}\right] = \left[\mathrm{ML^2T^{-2}}\right],$$

where [F] is dimension of force, so $\left[\mathrm{L^2T^{-2}}\right] = [\mathrm{EM^{-1}}]$ and therefore

$$[R_d] = \left[\frac{\mathrm{L^2}}{\mathrm{T^2}\Theta}\right] = \left[\mathrm{EM^{-1}}\Theta^{-1}\right],$$

which corresponds to the R_d 's SI unit of J kg^{-1} K^{-1}. The value R_d is approximately 287 of these units, that is $R_d \approx 287 \text{ J kg}^{-1} \text{K}^{-1}$ (we will come across this value repeatedly in this class).

Using this value of R_d, let us estimate air density ρ for some near-sea-level atmospheric conditions represented by p = 1000 hPa = 10^5 Pa = 10^5 kg s^{-2} m^{-1} and T = 293 K. We get

$$\rho = \frac{p}{R_d T} = \frac{10^5 \text{ kg s}^{-2} \text{ m}^{-1}}{287 \text{ J kg}^{-1} \text{K}^{-1} 293 \text{K}} = \frac{10^5}{287 \ \cdot 293} \text{kg m}^{-3} \approx 1.2 \text{ kg m}^{-3} .$$

3. Example of unit conversion in meteorology

Problem

During a passage of a low-pressure weather system through Oklahoma, atmospheric pressure measured at a Mesonet station was dropping at the rate of 10 hPa day^{-1}. Estimate this pressure drop rate in millimeters of mercury column per hour. Adopt density of mercury (Hg) to be $\rho_m = 13.5 \times 10^3 \text{ kg m}^{-3}$.

Solution

The pressure that a one-millimeter-high column of mercury exerts on a horizontal surface is

$$\text{mm}_{\text{Hg}} \equiv p_m = g \rho_m h_m = 9.8 \cdot 13500 \cdot 10^{-3} \approx 132 \text{ Pa} \simeq 1.3 \text{ hPa} .$$

Then the pressure drop rate of 10 hPa day^{-1} would correspond to

$$10 \text{ hPa day}^{-1} = \frac{10}{1.3} \text{ mm}_{\text{Hg}} \text{day}^{-1} = \frac{10}{1.3 \cdot 24} \text{ mm}_{\text{Hg}} \text{h}^{-1} \approx 0.3 \text{ mm}_{\text{Hg}} \text{h}^{-1} .$$