Lecture 30. November 7, 2016

Topics: Alternative forms of the first law of thermodynamics (FLT). Adiabatic process. Isentropic motion. Dry adiabatic lapse rate. Thermodynamic energy equation for synoptic-scale processes.

Reading: Chapter 2 in Holton and Hakim.

1. Alternative forms of the first law of thermodynamics (FLT)

Previously, the following expression of the FLT applied to a moving air parcel (also called the *thermodynamic energy equation*) has been derived:

$$\frac{de}{dt} + p\frac{d\alpha}{dt} = c_v\frac{dT}{dt} + p\frac{d\alpha}{dt} = J \ ,$$

where $e = c_v T$ is internal energy of air per unit mass, c_v is the specific heat at constant volume (for dry atmospheric air $c_v = 717$ J kg^{-1} K^{-1}), and $\alpha = \dfrac{1}{\rho}$ is the specific volume.

Using the equation of state of the dry atmospheric air (gas law),

$$p = \rho RT = \frac{1}{\alpha}RT \ ,$$

the derived equation may be rewritten as

$$p\frac{d\alpha}{dt} + \alpha\frac{dp}{dt} = R\frac{dT}{dt} \ ,$$

so the first law of thermodynamics (thermodynamic energy equation) takes the form:

$$(c_v + R)\frac{dT}{dt} - \alpha\frac{dp}{dt} = c_p\frac{dT}{dt} - \alpha\frac{dp}{dt} = J \ ,$$

or, again using the gas law,

$$c_p\frac{d\ln T}{dt} - R\frac{d\ln p}{dt} = \frac{J}{T} \ ,$$

where $c_p = c_v + R = 1004$ J kg^{-1} K^{-1} is (as follows from the above equation) the specific heat of dry air at constant pressure.

Thus, we have considered two forms of the FLT relevant for atmospheric processes,

$$c_v\frac{dT}{dt} + p\frac{d\alpha}{dt} = J \ ,$$

and

$$c_p\frac{dT}{dt} - \alpha\frac{dp}{dt} = J \ .$$

Now recall that, according to hydrostatic law, $\frac{dp}{dz} = -\rho g$, or $dp = -\rho g dz = -\rho d\Phi$, where Φ is geopotential. Therefore,

$$c_p \frac{dT}{dt} + \frac{d\Phi}{dt} = J \ ,$$

which provides another form of the FLT commonly considered in meteorology

Introducing

$$\frac{ds}{dt} \equiv \frac{J}{T} = \frac{1}{T}\frac{dq}{dt},$$

where *s* the *entropy* per unit mass and *dq* is the incremental change of thermal energy (heat) per unit mass over the time increment *dt*, we obtain for a reversible process (see Atmospheric Thermodynamics class):

$$c_p \frac{d \ln T}{dt} - R \frac{d \ln p}{dt} = \frac{ds}{dt}.$$

Taking the logarithm of the expression for *potential temperature*

$$\theta = T(p_0 / p)^{R/c_p},$$

which is the temperature that a parcel of dry air would have if it were expanded or compressed (or transformed in some other way) adiabatically to a standard pressure p_0 that is usually taken equal to 100kPa=1000hPa=1000mb, and differentiating the resulting expression, one can write

$$\ln\theta = \ln T + \frac{R}{c_p}\ln p_0 - \frac{R}{c_p}\ln p,$$

$$c_p d\ln\theta = c_p d\ln T - Rd\ln p = ds,$$

so the change of entropy is related to the change of potential temperature as

$$ds \equiv \frac{dq}{T} = \frac{Jdt}{T} = c_p d\ln\theta,$$

and a process, in which potential temperature is conserved, is also an isentropic (constant-entropy) process.

It should be reminded that $J = \frac{dq}{dt}$ in the above consideration is not a total derivative (like *dq* is not a perfect differential because heat or thermal energy is not a field variable, see the textbook of Wallace and Hobbs for additional explanations).

2. Adiabatic process; isentropic motion

In the absence of diabatic heating (or cooling), that is for an *adiabatic process* with $J = 0$ (in Class 32 we defined *J* as amount of thermal energy received or given away by a unit mass of the fluid element per unit time due to *diabatic* heating/cooling), the equations from p. **1** provide

$$c_p \ d\ln T - R\ d\ln p = 0, \text{ or } c_p d\ln\theta = 0, \text{ or } d\theta = 0, \text{ or } \theta = \text{const.}$$

Thus, in atmospheric motions that are (dry) adiabatic, the potential temperature is a conserved quantity in a sense that $d\theta$ =0.

Expressing

$$c_p \frac{d\ln\theta}{dt} = c_p \frac{d\ln T}{dt} - R\frac{d\ln p}{dt},$$

and recalling definition of the entropy (p. **1**), one can write

$$c_p \frac{d\ln\theta}{dt} = \frac{ds}{dt},$$

which means that a parcel, which moves conserving potential temperature also conserves the entropy. Motion along a constant θ surface is therefore called *isentropic* and the surface is also called *isentropic*.

3. Dry adiabatic lapse rate

Differentiating the expression for potential temperature over *z*, and then using the equation of state and the hydrostatic balance equation, we come to

$$c_p d\ln\theta = c_p d\ln T - Rd\ln p,$$

$$\frac{1}{\theta}\frac{d\theta}{dz} = \frac{1}{T}\frac{dT}{dz} - \frac{R}{c_p}\frac{1}{p}\frac{dp}{dz},$$

$$\frac{T}{\theta}\frac{d\theta}{dz} = \frac{dT}{dz} - \frac{R}{c_p}\frac{T}{\rho RT}\frac{dp}{dz} = \frac{dT}{dz} + \frac{1}{c_p}\frac{1}{\rho}\rho g,$$

$$\frac{T}{\theta}\frac{d\theta}{dz} = \frac{dT}{dz} + \frac{g}{c_p}.$$

So, in the adiabatic atmosphere with $\frac{d\theta}{dz} = 0$, the *lapse rate* of temperature (that is the rate of temperature decrease with height) is given by

$$-\frac{dT}{dz} = \frac{g}{c_p} \equiv \Gamma_d,$$

which is called the *dry adiabatic* (temperature) *lapse rate*.

4. Thermodynamic energy equation for synoptic-scale processes

Analogously to pressure and density fields (Class 29), we may decompose the total potential temperature field θ into a reference-state potential temperature that depends only on *z*: $\theta_r = \theta_r(z)$ and a deviation $\theta' = \theta'(x,y,z,t)$ from this basic-state value that depends on all coordinate directions and time:

$$\theta = \theta_r(z) + \theta'(x,y,z,t).$$

We now substitute $\theta = \theta_r(z) + \theta'(x,y,z,t)$ into the thermodynamic energy equation (see pp. **1** and **2**):

$$c_p \frac{d \ln \theta}{dt} = \frac{J}{T},$$

where we adopt an assumption $|\theta' / \theta_r| \ll 1$, which is generally valid for atmospheric motions on synoptic scales. We then arrive at the following approximate form of the heat balance (thermodynamic energy) equation:

$$\frac{1}{\theta_r}\left(\frac{\partial \theta'}{\partial t} + \mathbf{V} \cdot \nabla \theta'\right) + w \frac{d \ln \theta_r}{dz} = \frac{J}{c_p T},$$

where $\mathbf{V} \equiv (u, v)$ is the horizontal velocity vector.

One may further conclude that in the absence of strong diabatic heating (which is typically true outside regions of active precipitation and cloud layers), the heat balance in the atmosphere on the synoptic scales of motion is approximately given by

$$\frac{\partial \theta'}{\partial t} + \mathbf{V} \cdot \nabla \theta' + w \frac{d\theta_r}{dz} = 0.$$

Because in the synoptic-scale processes $\theta' \simeq \frac{\theta_r}{T_r} T'$, where T_r and T' are introduced through $T = T_r(z) + T'(x, y, z, t)$, the obtained approximate thermal energy balance equation may be rewritten also in terms of temperature deviation from the reference state as

$$\frac{\partial T'}{\partial t} + \mathbf{V} \cdot \nabla T' + w(\Gamma_d - \Gamma_r) = 0,$$

where $\Gamma_d = \frac{g}{c_p}$ is the dry adiabatic temperature lapse rate and $\Gamma_r = -\frac{dT_r}{dz}$ is the reference-state temperature lapse rate.