Lecture 31. November 9, 2016

Topics: Vector form of time-tendency and advection terms in momentum balance equations. Relation between local and total/substantial change of pressure in time. Autoconvection temperature lapse rate.

Reading: Chapters 1 and 2 of Holton and Hakim.

1. Vector form of time-tendency and advection terms in momentum balance equations

In the equations of motion with neglected curvature effects, the total derivative of velocity $\mathbf{U}(x,y,z,t)=\begin{pmatrix} u \\ v \\ w \end{pmatrix}$, including time-tendency and advection terms, has the following component form:

$$\frac{du}{dt}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\,,$$

$$\frac{dv}{dt}=\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\,,$$

$$\frac{dw}{dt}=\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\,.$$

How could the above relationships be expressed in vector form?

First, write down the velocity vector as

$$\mathbf{U}=u\mathbf{i}+v\mathbf{j}+w\mathbf{k}\,.$$

Then recall that

$$\nabla=\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\,,$$

which means that

$$\mathbf{U}\cdot\nabla=(u\mathbf{i}+v\mathbf{j}+w\mathbf{k})\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)=u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\,.$$

Now apply $\mathbf{U}\cdot\nabla$ to $\mathbf{U}$ as

$$(\mathbf{U}\cdot\nabla)\mathbf{U}=\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)(u\mathbf{i}+v\mathbf{j}+w\mathbf{k})=\begin{pmatrix} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} \\ u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} \\ u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} \end{pmatrix},$$

where on the right-hand side we have terms representing advection of individual velocity components.

On the other hand, one may also write the partial time derivative terms in vector form as

$$\frac{\partial u}{\partial t}\mathbf{i}+\frac{\partial v}{\partial t}\mathbf{j}+\frac{\partial w}{\partial t}\mathbf{k}=\frac{\partial \mathbf{U}}{\partial t}.$$

Collecting the time derivative and advection terms, we come to

$$\begin{pmatrix} \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z} \end{pmatrix}=\frac{\partial \mathbf{U}}{\partial t}+(\mathbf{U}\cdot\nabla)\mathbf{U}=\frac{d\mathbf{U}}{dt}.$$

2. Relation between local and total/substantial change of pressure in time

Suppose you are on a ship that is moving northward at a rate 10 km h^{-1}. The surface pressure increases toward the northwest at the rate of 5 Pa km^{-1}. What is the pressure tendency recorded at a nearby island station if the pressure aboard the ship decreases at a rate of 33.3 Pa h^{-1}? (This problem is a slightly paraphrased Problem 2.1 from H&H).

The ship is moving in space, so pressure tendency measured aboard the ship would correspond to the total change of pressure in time, *dp*/*dt*, as compared to the local change of pressure, $\partial p/\partial t$, measured at the stationary island. Therefore:

$$\frac{dp}{dt}=\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y},$$

where *u* and *v* are components of the ship motion in *x* and *y* directions, respectively, and there is no motion in the vertical, *z*, direction. The above relationship may be rewritten as

$$\frac{\partial p}{\partial t}=\frac{dp}{dt}-\mathbf{V}\cdot\nabla_h p,$$

where $\mathbf{V}=(u,v)$ is the vector of the ship's horizontal motion and $\nabla_h p$ is the vector of horizontal pressure gradient.

The ship is moving to the north, therefore vector $\mathbf{V}$ is directed northward. The horizontal pressure increase happens in the north-west direction, so vector $\nabla_h p$ is directed at the angle α=45º left of north. This provides for the $\mathbf{V}\cdot\nabla_h p$ term:

$$\mathbf{V}\cdot\nabla_h p=|\mathbf{V}||\nabla_h p|\cos\alpha=V|\nabla_h p|\cos\alpha,$$

where V = 10 km h^{-1} is the speed of the ship.

Knowing $\frac{dp}{dt}$ = – 33.3 Pa h^{-1} (on the ship) and $|\nabla_h p|$ = 5 Pa km^{-1}, we can evaluate $\frac{\partial p}{\partial t}$ on the island as

$$\frac{\partial p}{\partial t} = \frac{dp}{dt} - V\left|\nabla_h p\right|\cos\alpha = -33.3 - 10 \cdot 5 \cdot \sqrt{2}/2 \approx -70 \text{ Pa h}^{-1}.$$

3. Autoconvection temperature lapse rate

Assume ideal gas law and hydrostatic atmosphere. Which temperature lapse rate in such atmosphere corresponds to the zero vertical gradient of density ($d\rho / dz = 0$)? Derive an expression for the lapse rate in terms of known atmospheric parameters and calculate its value (naming variables that enter it) and a value (with units).

From $dp/dz = -\rho g$ (hydrostatic balance) and $p = \rho RT$ (ideal gas law), we have

$$\ln p = \ln \rho + \ln R + \ln T,$$

$$\frac{1}{p}\frac{dp}{dz} = \frac{1}{\rho}\frac{d\rho}{dz} + \frac{1}{T}\frac{dT}{dz},$$

$$\frac{dT}{dz} = \frac{T}{p}\frac{dp}{dz} = -\frac{T}{\rho RT}\rho g = -\frac{g}{R},$$

so the corresponding temperature lapse rate (which is called the *autoconvection lapse rate*) is

$$\Gamma = -\frac{dT}{dz} = \frac{g}{R} \approx 0.034 \text{ K m}^{-1},$$

where g is gravity (acceleration) and R is gas constant for (dry) air.

4. Geostrophic relationship in vector form

If only pressure-gradient and Coriolis terms are kept in the equations of the horizontal motion, they reduce to the geostrophic approximation

$$-fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \quad fu = -\frac{1}{\rho}\frac{\partial p}{\partial y},$$

where $f = 2\Omega \sin\varphi$ is the Coriolis parameter. The wind vector in this approximation (geostrophic wind) has only horizontal components and is given by

$$\mathbf{V}_g = \mathbf{i}u_g + \mathbf{j}v_g,$$

where

$$u_g = -\frac{1}{\rho f}\frac{\partial p}{\partial y} \text{ and } v_g = \frac{1}{\rho f}\frac{\partial p}{\partial x}$$

are the x and y components of the geostrophic wind.

How to show that in vector form the geostrophic wind may be written as

$$\mathbf{V}_g = \mathbf{k} \times \frac{1}{\rho f} \nabla p \,,$$

where del operator is applied on the horizontal (*X*-*Y*) plane, so $\nabla p = \frac{\partial p}{\partial x}\mathbf{i} + \frac{\partial p}{\partial x}\mathbf{j}$?

First, multiply ∇p by $\frac{1}{\rho f}$ to get

$$\frac{1}{\rho f}\nabla p = \frac{1}{\rho f}\left(\frac{\partial p}{\partial x}\mathbf{i} + \frac{\partial p}{\partial y}\mathbf{j}\right) = \frac{1}{\rho f}\frac{\partial p}{\partial x}\mathbf{i} + \frac{1}{\rho f}\frac{\partial p}{\partial y}\mathbf{j} = v_g\mathbf{i} - u_g\mathbf{j} \,.$$

The right-hand side of this expression looks like a vector somehow related to the geostrophic wind, but it is definitely not what we want. However, apply $\mathbf{k}\times$ to both sides of the obtained equation:

$$\mathbf{k} \times \frac{1}{\rho f}\nabla p = \frac{1}{\rho f}\left(\frac{\partial p}{\partial x}\mathbf{k}\times\mathbf{i} + \frac{\partial p}{\partial y}\mathbf{k}\times\mathbf{j}\right) = \frac{1}{\rho f}\frac{\partial p}{\partial x}\mathbf{k}\times\mathbf{i} + \frac{1}{\rho f}\frac{\partial p}{\partial x}\mathbf{k}\times\mathbf{j} \,,$$

and use the right-hand rule to proceed to

$$\mathbf{k} \times \frac{1}{\rho f}\nabla p = \frac{1}{\rho f}\frac{\partial p}{\partial x}\mathbf{j} - \frac{1}{\rho f}\frac{\partial p}{\partial y}\mathbf{i} \,.$$

Rearrange the terms and substitute $u_g = -\frac{1}{\rho f}\frac{\partial p}{\partial y}$ and $v_g = \frac{1}{\rho f}\frac{\partial p}{\partial x}$ to get

$$\mathbf{k} \times \frac{1}{\rho f}\nabla p = -\frac{1}{\rho f}\frac{\partial p}{\partial y}\mathbf{i} + \frac{1}{\rho f}\frac{\partial p}{\partial x}\mathbf{j} = u_g\mathbf{i} + v_g\mathbf{j} = \mathbf{V}_g \,.$$