Lecture 33. November 14, 2016

Topics: Vertical acceleration of an air parcel. Buoyancy. Buoyancy frequency. Static stability criteria for dry air.

Reading: Holton and Hakim Chapter 2

1. Vertical acceleration of an air parcel due to density difference

Let us consider force per unit mass (acceleration) acting in the vertical on an air parcel with density ρ surrounded by air with reference density ρ_r and pressure p_r in a hydrostatic equilibrium given by the hydrostatic balance equation,

$$\frac{1}{\rho_r}\frac{\partial p_r}{\partial z} = -g \text{ .}$$

However, the parcel, whose density and pressure are ρ and pressure p, respectively, is not in the hydrostatic equilibrium, so it will experience the vertical acceleration, which, according to the third equation of motion, will be

$$\frac{dw}{dt} = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} \text{ .}$$

For relatively slow motions of the parcel with respect to speed of sound, one may assume that pressure in the parcel instantaneously (in practical terms) adjusts to the pressure of the ambient air, so $\frac{\partial p}{\partial z}$ is (roughly) equal to $\frac{\partial p_r}{\partial z}$, and therefore we can write down for the parcel acceleration:

$$\frac{dw}{dt} = -g - \frac{1}{\rho}\frac{\partial p_r}{\partial z} = -g + \frac{\rho_r}{\rho}g = -g\frac{\rho - \rho_r}{\rho} \text{ ,}$$

which means that a parcel with density larger that the density of the ambient air $\rho > \rho_r$ will accelerate downwards ($\frac{dw}{dt} < 0$) and a parcel with density smaller that the density of the ambient air $\rho < \rho_r$ will accelerate upwards ($\frac{dw}{dt} > 0$).

2. Buoyancy

The following quantity with dimension of acceleration (m s^{-2} in SI units):

$$b = -g\frac{\rho - \rho_r}{\rho} \text{ ,}$$

which (as follows from p. **1**) represents the vertical acceleration of an air parcel associated with the Archimedes force, is called the *buoyancy*.

In atmospheric dynamics, the buoyancy is usually expressed in terms of potential temperature

$$\theta = T(p_0 / p)^{R/c_p} .$$

To derive this expression, use the equation of state $p = \rho RT$ and the approximation $p = p_r$ (the pressure in the parcel instantaneously adjusts to the pressure of the ambient air, see p. **1**). By expressing the air parcel and ambient densities as

$$\rho = \frac{p_r}{RT} \text{ and } \rho_r = \frac{p_r}{RT_r} ,$$

we come first to

$$b = g\frac{T - T_r}{T_r} ,$$

and then to

$$b = g\frac{\theta - \theta_r}{\theta_r} ,$$

where

$$\theta_r = T_r(p_0 / p_r)^{R/c_p} .$$

is the reference (environmental) value of the potential temperature.

Within the framework of the Boussinesq approximation (it was introduced and discussed in Class 30), the buoyancy is expressed as

$$b = g\frac{\theta - \theta_r}{\theta_c} ,$$

where θ_c is a constant reference value of the potential temperature. Note that generally $\theta_c \neq \theta_r$, but the environmental potential temperature θ_r may be chosen to be equal to θ_c.

3. Buoyancy frequency

Assuming that the reference potential temperature is a function of *z* only, $\theta_r = \theta_r(z)$ we may fix the origin of *z* axis at some level in the atmosphere and expand the vertical profile of the reference potential temperature into Taylor series for small *z* around 0 keeping only linear terms:

$$\theta_r(z) = \theta_r(0) + \frac{d\theta_r}{dz} z .$$

If the parcel starts its vertical motion at level *z*=0, where it was in balance with the environmental air, $\theta(0) = \theta_r(0)$, and moves adiabatically, which means that it conserves its potential temperature, $\theta(z) = \theta_r(0)$ (see p. **2**), so the above expression may be rewritten as

$$\theta_r(z) = \theta(z) + \frac{d\theta_r}{dz} z \, ,$$

that is

$$\theta - \theta_r = -\frac{d\theta_r}{dz} z \, ,$$

and therefore

$$\frac{dw}{dt} = \frac{d^2 z}{dt^2} = b = -\frac{g}{\theta_r}\frac{d\theta_r}{dz} z = -N^2 z \, ,$$

where

$$N \equiv \sqrt{\frac{g}{\theta_r}\frac{d\theta_r}{dz}}$$

is the so-called *buoyancy frequency* (also known as the Brunt-Väisälä frequency).

4. Static stability criteria for dry air

The following quantity associated with Brunt-Väisälä (buoyancy) frequency,

$$N^2 = \frac{g}{\theta_r}\frac{d\theta_r}{dz} = g\frac{d\ln\theta_r}{dz}$$

is commonly used a measure of static stability in the atmosphere.

If $\frac{d\theta_r}{dz} > 0 \Rightarrow N^2 > 0$, the displaced parcel will be forced back to its initial position by the Archimedes force. This situation corresponds to statically ***stable*** conditions in the atmosphere.

If $\frac{d\theta_r}{dz} < 0 \Rightarrow N^2 < 0$, the displaced parcel will be further accelerated in the direction of displacement by the Archimedes force. This situation corresponds to statically ***unstable*** conditions in the atmosphere.

If $\frac{d\theta_r}{dz} = 0 \Rightarrow N^2 = 0$, the displaced parcel will be in the equilibrium with environmental air at the level to which it was displaced. This corresponds to statically ***neutral*** conditions in the atmosphere.