Lecture 35. November 21, 2016

Topics: Natural coordinates. Balanced flow. Particular cases of balanced flow. Gradient wind. Relation between gradient and geostrophic winds. Cyclonic and anticyclonic circulations in natural coordinates.

Reading: Holton and Hakim Section 3.2.

1. Equation of horizontal momentum balance in natural coordinates

In Class 37 we obtained the following momentum balance equation for the horizontal atmospheric flow in isobaric coordinates:

$$\frac{d\mathbf{V}}{dt} + f\mathbf{k}\times\mathbf{V} = -\nabla_p \Phi .$$

Now we consider this flow in the so-called *natural coordinate system* with orthogonal unit vectors $\mathbf{t}$ and $\mathbf{n}$ whose directions are associated with local direction of the horizontal velocity $\mathbf{V}$, where $\mathbf{V} = V\mathbf{t}$, $V = \frac{ds}{dt}$, and $s=s(x, y, t)$ is the distance along the way of the parcel moving on the horizontal plane, while the unit coordinate vector **n** is taken normal to $\mathbf{V}$ and directed left of $\mathbf{V}$ (see Fig. 3.1 in the textbook). Since n is the distance in the **n** direction, n is positive to the left of the flow direction.

Let us express $\frac{d\mathbf{V}}{dt}$ in such natural coordinates s and n on the isobaric surface, taking into account that unit vectors associated with these coordinate directions are $\mathbf{t}$ and $\mathbf{n}$, respectively. Calculating the time derivative of $\mathbf{V}$, we obtain

$$\frac{d\mathbf{V}}{dt} = \frac{dV\mathbf{t}}{dt} = \mathbf{t}\frac{dV}{dt} + V\frac{d\mathbf{t}}{dt}.$$

Consider motion of an air parcel along a distance (arc) element δs over a curved path with the curvature radius R (Fig. 3.1 in the textbook). In the process of such motion, vector $\mathbf{t}$ will change to the vector $\mathbf{t}+\delta\mathbf{t}$ directed at the angle $\delta\psi$ to $\mathbf{t}$. Note, though, that $|\mathbf{t}+\delta\mathbf{t}| = |\mathbf{t}| = 1$. Based on geometrical considerations, $\delta\psi$ equals to the arc angle of δs. Because angle $\delta\psi$ is supposed to be small, so that $\delta\psi \simeq \sin\delta\psi \simeq \tan\delta\psi$, we can write

$$\delta\psi = \frac{\delta s}{|R|} = \frac{|\delta\mathbf{t}|}{|\mathbf{t}|} = |\delta\mathbf{t}|,$$

where the absolute value of the curvature radius R is used because it may be conventionally either positive or negative.

It is taken positive when the center of curvature is in the positive $\mathbf{n}$ direction (as it is in Fig. 3.1 from the textbook). Therefore, with $R > 0$, the moving air parcel turns toward left following the motion, and with $R < 0$, the parcel turns right following the motion.

In the limit of $\delta s \to 0$, $\delta \mathbf{t}$ would point to the curvature center, that is in $\mathbf{n}$ direction (along *R*). Rewriting the previous relationship taking this direction into account, we have

$$\frac{|\delta \mathbf{t}|}{\delta s} = \frac{1}{R},$$

or

$$\frac{|\delta \mathbf{t}|\mathbf{n}}{\delta s} = \frac{\delta \mathbf{t}}{\delta s} = \frac{\mathbf{n}}{R},$$

or, switching to the differential form,

$$\frac{d\mathbf{t}}{ds} = \frac{\mathbf{n}}{R}.$$

Taking into account that $V = \frac{ds}{dt}$, we can now rewrite the above expression in terms of the time differential dt as

$$\frac{d\mathbf{t}}{dt} = \frac{d\mathbf{t}}{ds}\frac{ds}{dt} = \frac{\mathbf{n}}{R}\frac{ds}{dt} = \frac{\mathbf{n}}{R}V,$$

Using this result in the equation for $\frac{d\mathbf{V}}{dt}$, we come to

$$\frac{d\mathbf{V}}{dt} = \mathbf{t}\frac{dV}{dt} + \mathbf{n}\frac{V^2}{R}.$$

Now we need to take care about the Coriolis term $f\mathbf{k} \times \mathbf{V}$. Using the right-hand rule, we find that vector $\mathbf{k} \times \mathbf{V}$ is directed along $\mathbf{n}$. On the other hand, its magnitude is *V*. Therefore,

$$f\mathbf{k} \times \mathbf{V} = fV\mathbf{n}.$$

As for the gradient of geopotential, $\nabla_p \Phi$, in the natural coordinates *s* and *n* it appears as

$$\nabla_p \Phi = \mathbf{t}\frac{\partial \Phi}{\partial s} + \mathbf{n}\frac{\partial \Phi}{\partial n}.$$

Therefore, the horizontal momentum equation in the natural coordinates has the following form:

$$\mathbf{t}\frac{dV}{dt} + \mathbf{n}\frac{V^2}{R} = -\mathbf{n}fV - \mathbf{t}\frac{\partial \Phi}{\partial s} - \mathbf{n}\frac{\partial \Phi}{\partial n},$$

which can be written as two equations for *s* and *n* components of momentum:

$$\frac{dV}{dt} = -\frac{\partial \Phi}{\partial s},$$

$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}.$$

A steady flow that obeys

$$\frac{dV}{dt} = -\frac{\partial \Phi}{\partial s} = 0$$

and satisfies the second of the above equations,

$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n},$$

with R = const and $\frac{\partial \Phi}{\partial n}$ = const is called the *balanced flow*.

2. Particular cases of balanced flow

Geostrophic flow

This balanced flow case corresponds to the case when $R \to \pm\infty$. This results in

$$fV = -\frac{\partial \Phi}{\partial n},$$

which is an expression of geostrophic balance in the natural coordinates. The *geostrophic flow* velocity (geostrophic wind velocity, see Class 29) in natural coordinates is thus presented by

$$V_g = -\frac{1}{f}\frac{\partial \Phi}{\partial n}.$$

Inertial flow

In the absence of the pressure gradient force ($\frac{\partial \Phi}{\partial n} = 0$), the momentum equation in natural coordinates reduces to

$$\frac{V}{R} + f = 0.$$

The balanced flow case described by this equation is called the *inertial flow*. Its velocity is given by $V = -fR$.

The corresponding motion is circular, with radius $R = -V/f$ =const. It is *anticyclonic* (directed clockwise in the Northern Hemisphere and anticlockwise in the Southern Hemisphere for an external observer).

The period of this circular motion (also called the *inertial oscillation*, see Class 24) is

$$P = \left|\frac{2\pi R}{V}\right| = \frac{2\pi}{|f|}.$$

Cyclostrophic flow

This balanced flow case is realized in the case of small-scale horizontal motion when the Coriolis force can be neglected, which also implies that the ratio of the centrifugal force to the Coriolis force, expressed as the Rossby number of the flow, Ro=$\frac{V}{|fR|}$ (see Class 29), is large. For a typical tornado, for instance, the Coriolis force can be neglected in the force balance considerations. The force balance in such *cyclostrophic flow* is given by

$$\frac{V^2}{R} = -\frac{\partial \Phi}{\partial n},$$

and the corresponding wind velocity is

$$V = \left(-R \frac{\partial \Phi}{\partial n} \right)^{1/2}.$$

3. Gradient wind

An atmospheric flow that satisfies the balanced flow equation in the natural coordinates,

$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}.$$

is called the *gradient wind*. Accordingly, the *gradient wind speed* is given by

$$V = -\frac{fR}{2} \pm \left(\frac{f^2 R^2}{4} - R \frac{\partial \Phi}{\partial n} \right)^{1/2},$$

Recalling that the geostrophic balance in natural coordinates is given by

$$fV_g = -\frac{\partial \Phi}{\partial n},$$

one may write the gradient wind speed expression as

$$V = -\frac{fR}{2} \pm \left(\frac{f^2 R^2}{4} + fRV_g \right)^{1/2},$$

Note, that according to the way the gradient wind speed V was introduced, its value should be real and non-negative (because, in fact, it is the wind velocity vector *magnitude*, see p. **1**). The value of geostrophic velocity V_g, however, can be of any sign.

Using the definition of the geostrophic wind in natural coordinates, we have

$$\frac{V^2}{R} + fV - fV_g = 0,$$

which provides the following ratio between the gradient wind speed and geostrophic wind velocity:

$$\frac{V_g}{V} = 1 + \frac{V}{fR}.$$

4. Cyclonic and anticyclonic circulations

The gradient flow is *cyclonic* when $fR > 0$, so the direction of the circulation is in the same direction as the Earth's rotation for an observer placed in space. In this case (see p. **3**): $\frac{V_g}{V} > 1$.

The gradient flow is *anticyclonic* when $fR < 0$, so the direction of the circulation is opposite to the Earth rotation for an observer placed in space. In this case (see p. **3**): $\frac{V_g}{V} < 1$.

Note that these definitions of cyclonic and anticyclonic circulations are valid for both hemispheres.