Lecture 4. August 29, 2016

Topics: Purpose of a coordinate system on a line, plane, and in space. Cartesian coordinate systems in one, two, and three dimensions. History of the Cartesian system. Orientation of coordinate axes. Coordinate transformations. Intervals, lines, and vectors in Cartesian coordinates.

Reading: Appendix C.3 of Holton and Hakim. Section 3 of Fiedler.

1. Purpose of a coordinate system on a line, plane, and in space

Defining position of a point in a one-dimensional (1-D) space, that is along a straight line, starts with defining a point on the line from which distances are counted (this point is usually called the ***origin***) and a unit of length. The distance measured from the point is called the ***coordinate***.

A ***coordinate system*** on a plane (in two dimensions) uniquely specifies each point in a plane by a pair of numerical coordinates. One can use the same principle to specify the position of any point in three-dimensional space where three coordinates need to be used, or, more generally, in n-dimensional space with n coordinates.

2. Cartesian coordinate systems in one, two, and three dimensions

Cartesian coordinate system in 1-D space (on a line) specifies which of the two half-lines determined by the ***origin*** (usually denote as O) is used to count positive distances from the origin (so the second half-line would be used to count negative distances). This choice defines the orientation of the coordinate line: it points (or it is oriented) from the negative half towards the positive half. Then each point on the line can be specified by its distance from the origin O (coordinate), taken with a plus or minus sign depending on which half-line contains the point under consideration. A line with a chosen Cartesian system is called a ***number line***. Every real number, whether integer, rational, or irrational, has a unique location on the line. Conversely, every point on the line can be interpreted as a number in an ordered continuum which includes the real numbers.

The adjective ***Cartesian*** refers to the French mathematician and philosopher René Descartes (who used the name *Cartesius* in Latin).

The ***2-D Cartesian coordinate system*** on a plane, the coordinates are the signed distances from the point to two fixed perpendicular directed lines, measured in the same unit of length. Each reference line is called a ***coordinate axis*** or just ***axis*** of the system. The coordinates are usually denoted as x and y, or (x, y), and the corresponding axes are denoted as X and Y. The point where the axes meet is called ***origin*** (denoted, like in the case of the 1-D system as O). Coordinates of the origin are (0, 0). The coordinates can also be defined as the positions of the perpendicular projections of the point onto the two axes, expressed as signed distances from the origin.

The ***3-D Cartesian coordinate system*** in space (three dimensions) uses three mutually perpendicular coordinate axes (usually specified as *X*, *Y*, and *Z*) with coordinates denoted, respectively, as *x*, *y*, and *z*. Thus, a point in 3-D space has Cartesian coordinates (*x*, *y*, *z*).

3. History of the Cartesian system

The idea of the Cartesian system was developed in 1637 in writings by Descartes and independently by Pierre de Fermat (according to Wikipedia), although Fermat also worked in three dimensions, and did not publish the discovery. Both authors used a single axis in their treatments and have a variable length measured in reference to this axis. The concept of using a pair of axes was introduced in later work by commentators who were trying to clarify the ideas contained in Descartes' *La Géométrie*. The development of the Cartesian coordinate system would play an intrinsic role in the development of the calculus by Isaac Newton and Gottfried Wilhelm Leibniz. Nicole Oresme, a French cleric and friend of the dauphin (later to become King Charles V) of the 14th Century, used constructions similar to Cartesian coordinates well before the time of Descartes and Fermat. Many other coordinate systems have been developed since Descartes, such as the polar coordinates for the plane, and the spherical and cylindrical coordinates for three-dimensional space.

4. Orientation of coordinate axes

- Orientation of Cartesian axes in two dimensions.
- Orientation of Cartesian axes in three dimensions.
- Quadrants and octants.
- Right-hand rule.
- Right-handed (positive) and left-handed (negative) coordinate systems.

5. Coordinate transformations in the 2-D Cartesian system

- Translation:

$$(x', y') = (x + a, y + b).$$

- Rotation.

  Rotating coordinate system by an angle α counterclockwise transforms point (x, y) into (x', y') with the coordinates related as

$$x' = x\cos\alpha + y\sin\alpha,$$

$$y' = y\cos\alpha - x\sin\alpha,$$

  or

$$(x', y') = (x\cos\alpha + y\sin\alpha,\ y\cos\alpha - x\sin\alpha).$$

- Reflection.

Reflection across a line through the origin making an angle α with the X axis transforms point (x, y) into point (x', y') with the coordinates related as

$$x' = x\cos 2\alpha + y\sin 2\alpha,$$

$$y' = x\sin 2\alpha - y\cos 2\alpha,$$

or

$$(x', y') = (x\cos 2\alpha + y\sin 2\alpha,\ x\sin 2\alpha - y\cos 2\alpha).$$

- Scaling:

$$(x', y') = (mx, my).$$

6. Distances, lines, and position vectors in Cartesian coordinates.

- Distance between two points in the 2-D:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

and 3-D:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2},$$

coordinate systems.

- Distance to the origin of the system.

2-D:

$$d_0 = \sqrt{x^2 + y^2},$$

3-D:

$$d_0 = \sqrt{x^2 + y^2 + z^2},$$

- Coordinates of the middle point of an interval in the 2-D system:

$$x = \frac{x_2 + x_1}{2},\ y = \frac{y_2 + y_1}{2},$$

where (x_1, y_1) and (x_2, y_2) are, respectively, starting and ending coordinates of the interval.

- Equation of a straight line in the 2-D system.

General form:

$$Ax + By + C = 0,$$

(with $A, B \neq 0$ together).

Angle-shift form:

$$y = ax + b.$$

Axis-interval form:

$$\frac{x}{c} + \frac{y}{d} = 1.$$

- Equation of a circle of radius r in the 2-D system.

  Center of the circle is in the coordinate-system origin:

$$x^2 + y^2 = r^2 ,$$

  Center of the circle has coordinates (a, b):

$$(x-a)^2 + (y-b)^2 = r^2 .$$

- Position vector in the Cartesian system.

  A point in space in a Cartesian coordinate system may be represented by a *position vector* which is directed from the origin of the coordinate system to the point.

  2-D:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} ,$$

  where $\mathbf{i}, \mathbf{j}$ are the unit vectors along coordinate axes X and Y, respectively'

  3-D:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} ,$$

  where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors along coordinate axes X, Y, and Z, respectively.

  Unit vectors are defined as $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$, $\mathbf{k} = (0,0,1)$ and, considered together, referred to as the ***standard (Cartesian) basis***.

Cartesian coordinates in two dimensions

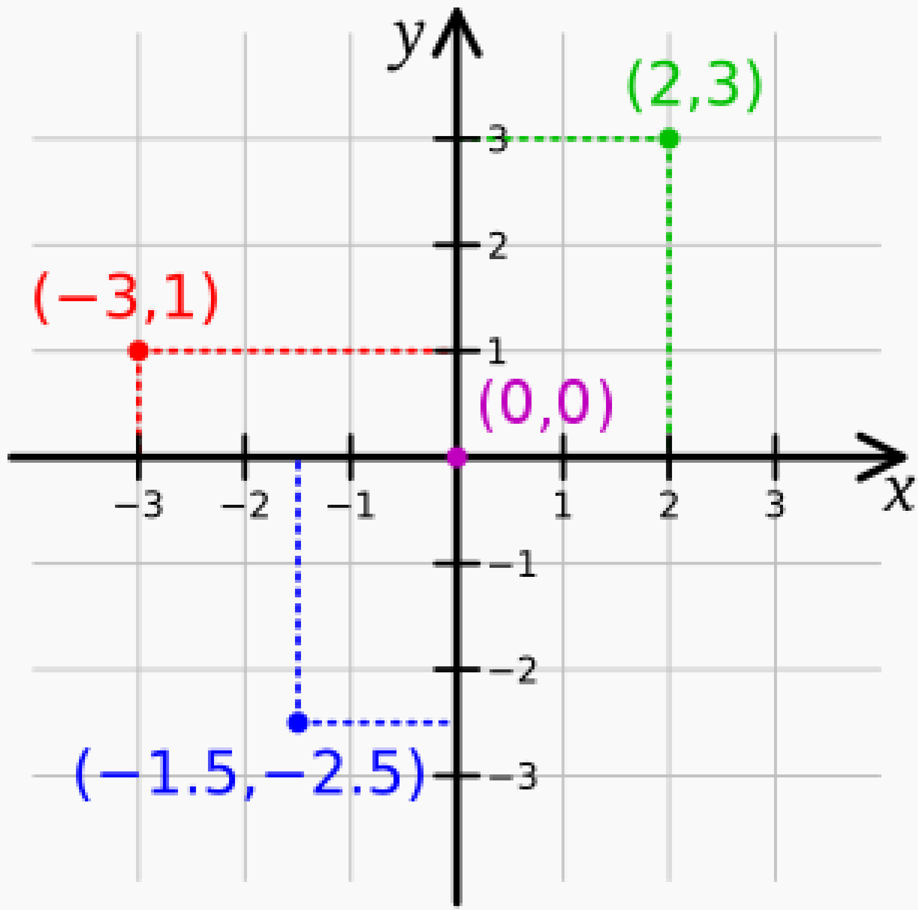
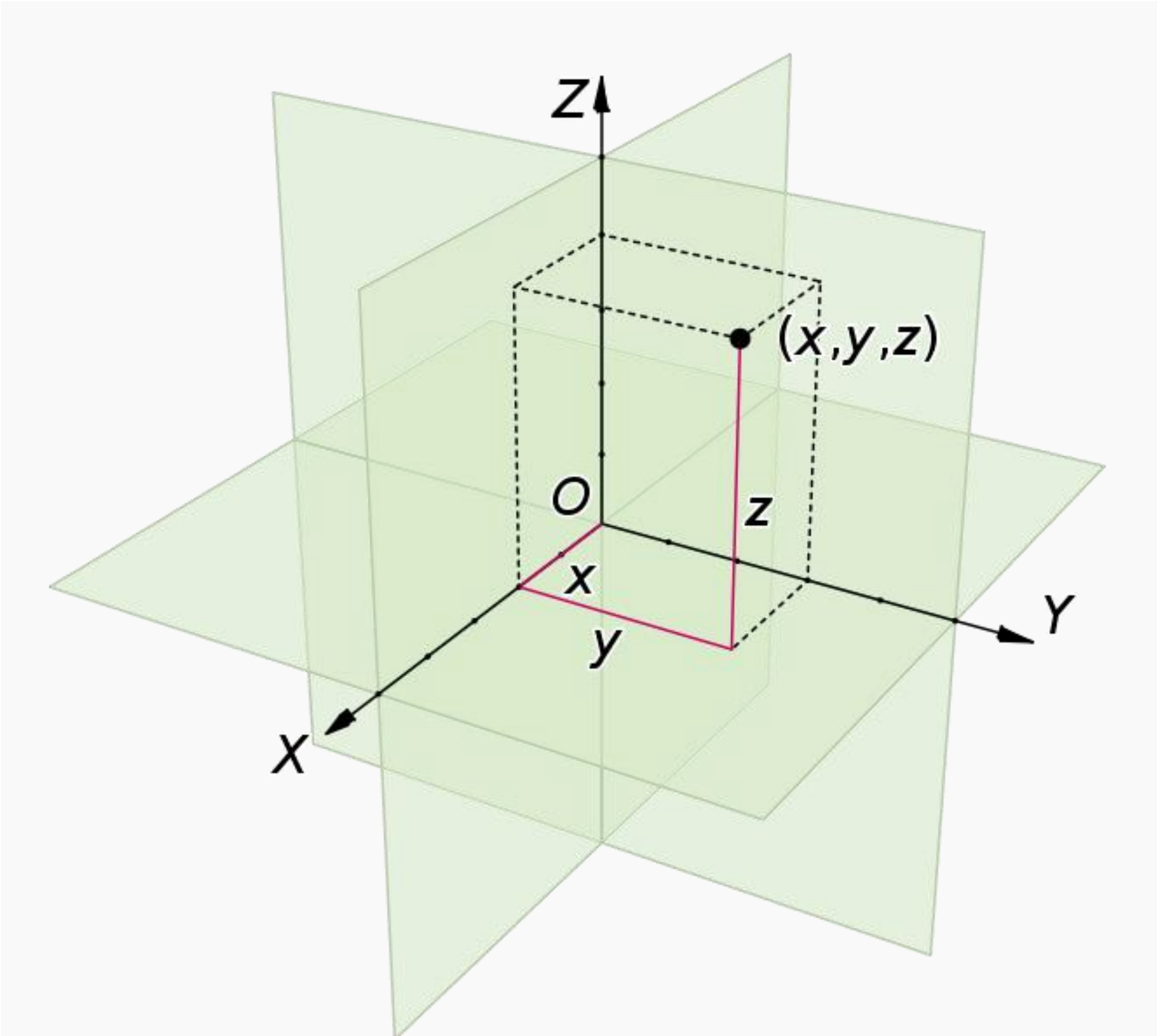


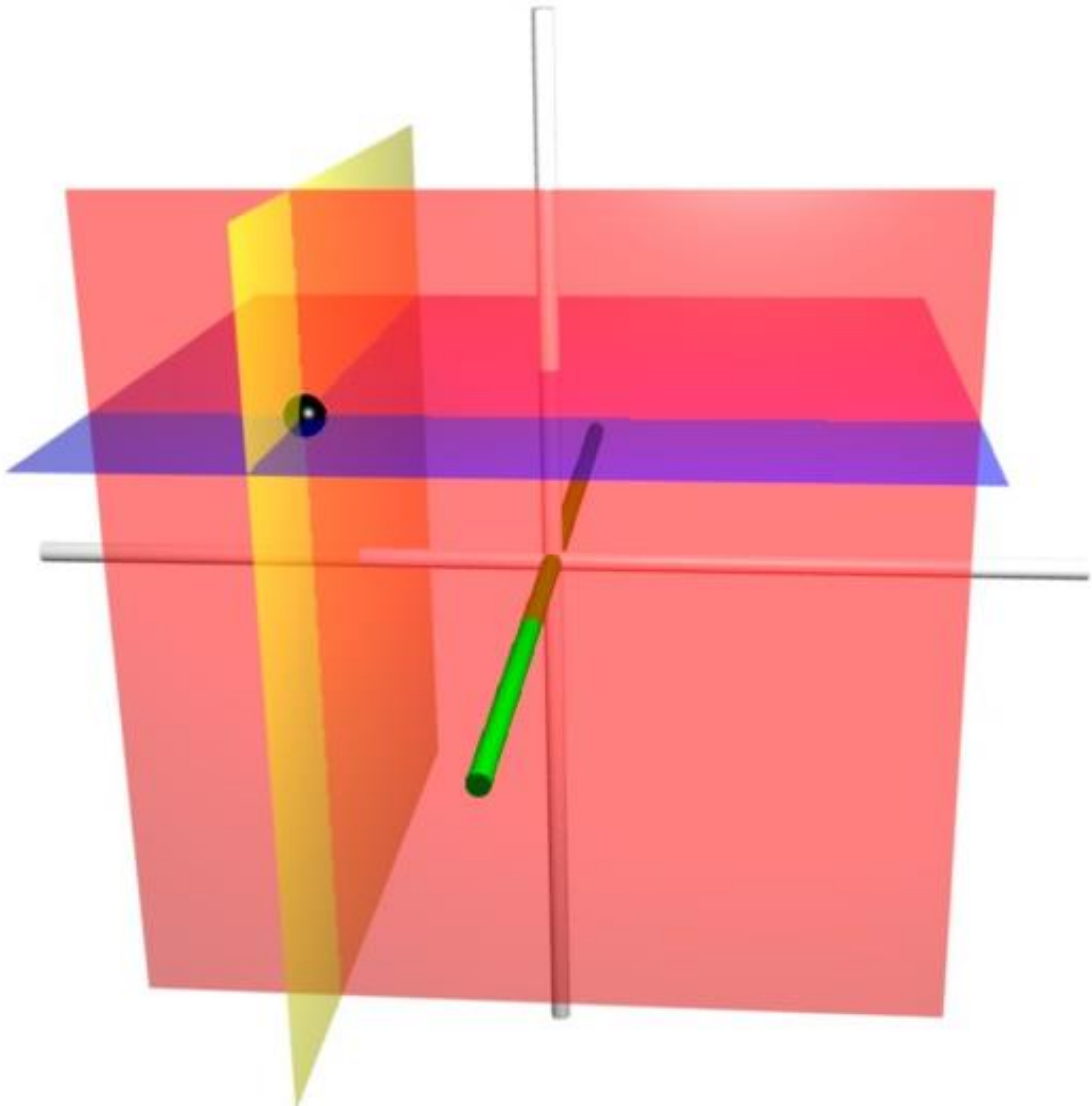
Illustration of a Cartesian coordinate plane with the following points marked and labeled with their coordinates: (2, 3) in green, (−3, 1) in red, (−1.5, −2.5) in blue, and the origin (0, 0) in purple.

Cartesian coordinates in three dimensions



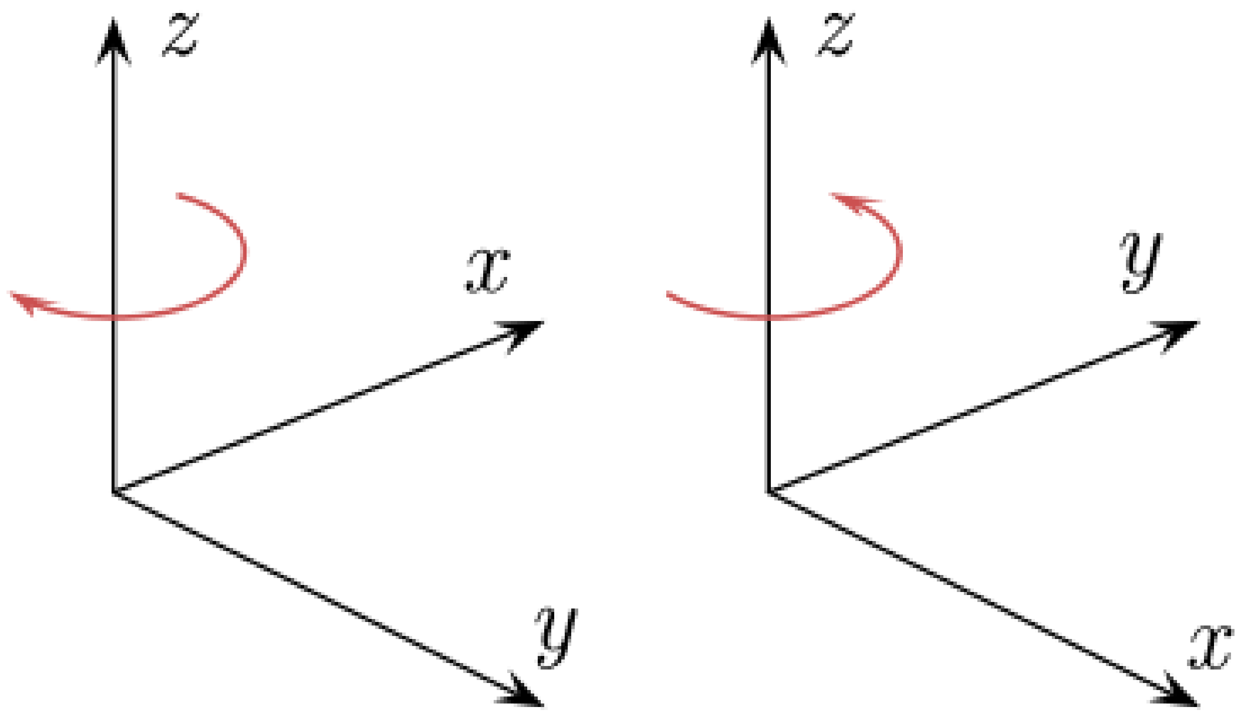
A three dimensional Cartesian coordinate system, with origin *O* and axis lines *X*, *Y* and *Z*, oriented as shown by the arrows.

Coordinate surfaces in Cartesian system



The coordinate surfaces of the Cartesian coordinates (x, y, z). The z-axis is vertical and the x-axis is highlighted in green. Thus, the red plane shows the points with x=1, the blue plane shows the points with z=1, and the yellow plane shows the points with y=−1. The three surfaces intersect at the point **P** (black sphere) with the Cartesian coordinates (1, −1, 1).

Cartesian system handedness



The left-handed orientation is on the left, and the right-handed orientation is on the right.

Coordinate planes in the right-handed Cartesian coordinate system