Lecture 5. August 31, 2016

Topics: Polar coordinate system. Conversion of polar coordinates to 2-D Cartesian coordinates. Cylindrical coordinate system. Spherical coordinate system. Conversion of spherical coordinates to Cartesian and cylindrical coordinates.

Reading: Appendix C.3 of Holton and Hakim. Sections 3, 9, and 10 of Fiedler.

1. Polar coordinate system

Polar coordinate system is a two-dimensional coordinate system (usually, on a geometrical plane) in which each point is determined by a distance from a fixed point and an angle from a fixed direction.

The following notations and conventions are adopted in the polar coordinate system:

- The fixed point (analogous to the origin in the Cartesian system, see Class 4) is called the *pole*, and the ray from the pole in the fixed direction is the *polar axis*.
- The distance from the pole is called the *radial coordinate* or simply *radius* (normally denoted as r), and the angle is the *angular coordinate*, *polar angle*, or *azimuth* (*azimuthal angle*), normally denoted as θ.
- In most contexts, a positive angular coordinate means that the polar angle θ is measured counterclockwise with respect to the polar axis.
- The polar angle is generally expressed in either degrees or radians (2π rad being equal to 360°).
- Because a unique representation is usually needed for any coordinate point, it is common to limit r to non-negative numbers ($r \geq 0$) and θ to the interval [0, 360°) or (−180°, 180°] (in radians, $[0, 2\pi)$ or $(-\pi, \pi]$).
- For uniqueness one may assign a fixed azimuth for the pole, e.g., $\theta = 0$ at the pole.

2. Conversion of polar coordinates to 2-D Cartesian coordinates

Polar coordinates r and θ are converted to the plane Cartesian coordinates x and y (discussed in Class 4) as

$$x = r\cos\theta \text{ and } y = r\sin\theta .$$

Inversely, Cartesian coordinates x and y can be converted to polar coordinates r and θ with $r \geq 0$ and θ in the interval $(-\pi, \pi]$ as

$$r = \sqrt{x^2 + y^2} \text{ and } \theta = \arctan(y / x) .$$

A circle of radius c, which in Cartesian coordinates appears as $x^2 + y^2 = c^2$ (for the case when the center of the circle is at the origin), or $(x-a)^2 + (y-b)^2 = c^2$ [for the case when center of the circle has coordinates (a, b)], see Class 4, in polar coordinates appears as

$$r(\theta) = c$$

for a circle with the origin at the pole, and

$$r^2 - 2rr_0\cos(\theta - \theta_0) + r_0^2 = c^2$$

for a circle with the origin at (r_0, θ_0).

While evaluating the value of the $\arctan(y/x)$ function, normally considered in the interval from −90° (−π/2) to 90° (π/2), the quadrant of the Cartesian coordinates should be appropriately taken into account. Specifically,

$$\theta = \begin{cases} \arctan\left(\dfrac{y}{x}\right) & \text{if } x > 0 \\ \arctan\left(\dfrac{y}{x}\right) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan\left(\dfrac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ \dfrac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\dfrac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \\ 0 & \text{if } x = 0 \text{ and } y = 0. \end{cases}$$

Relationships between derivatives in Cartesian and polar coordinates for a given function $u = u(x, y)$ are the following:

$$\frac{\partial u}{\partial r} = \frac{x}{r}\frac{\partial u}{\partial x} + \frac{y}{r}\frac{\partial u}{\partial y},$$

$$\frac{\partial u}{\partial \theta} = -y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y}.$$

Cartesian slope of the tangent line to a polar curve $r(\theta)$ at any given point is given by

$$\frac{dy}{dx} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

3. Cylindrical coordinate system

Cylindrical coordinate system is a 3-D coordinate system that specifies point positions by

(i) distance from a chosen reference axis,

(ii) direction from the axis relative to a chosen reference direction, and

(iii) distance from a chosen reference plane perpendicular to the axis.

The latter distance may be negative or positive depending on which side of the reference plane the point under consideration is placed.

- *Origin* of the system is the point where all three coordinates can be set to zero. This is the intersection between the reference plane and the axis.
- The axis is variously called the *cylindrical* or *longitudinal* axis, to differentiate it from the *polar axis*, which is the ray that lies in the reference plane, starting at the origin and pointing in the reference direction.
- Distance from the axis may be called the *radial distance* or *radius*, while the angular coordinate is sometimes referred to as the *angular position* or as the *azimuth*.
- Radius and azimuth are together called the *polar coordinates*, as they correspond to a 2-D polar coordinate system (see p. **1** and p. **2**) in the reference plane.
- The third coordinate may be called the *height* or *altitude* (if the reference plane is considered horizontal), *longitudinal position*, or *axial position*.

Rules of conversion of cylindrical coordinates to Cartesian coordinates are analogous to the conversion rules for the polar coordinates (with longitudinal the axis in cylindrical coordinates being aligned with one of Cartesian axes − typically with *Z*).

4. Spherical coordinate system

To define a spherical coordinate system, one must choose two orthogonal directions, the *zenith* and the *azimuth* directions, and an *origin* point in space. These choices determine a reference plane that contains the origin and is perpendicular to the zenith. The spherical coordinates of a point *P* are then defined in terms of:

- *Radius* or *radial distance* from the origin *O* to the point *P*.
- *Inclination* (or *zenith angle*, or *polar angle*) − the angle between the zenith direction and the line interval *OP*.
- *Azimuth* (or *azimuthal angle*) − signed angle measured from the azimuth reference direction to the orthogonal projection of the line interval *OP* on the reference plane.

The sign of the azimuth is determined by choice of positive sense of turning about the zenith direction. This choice is arbitrary, and is part of the coordinate system's definition. The *elevation angle* is 90 degrees ($\pi/2$ radians) minus the inclination angle. If the inclination is zero or 180 degrees (π radians), the azimuth is arbitrary. If the radius is zero, both azimuth and inclination are arbitrary.

For practical reasons, it is handy to define a unique set of spherical coordinates for each point that would eliminate non-uniqueness of the coordinate prescription.

Commonly chosen limits for radial distance r, inclination angle θ, and azimuthal angle φ are $r > 0$, $0 \le \theta \le 180°$ (π rad), and $0 \le \varphi < 360°$ (2π rad). Note that even with these restrictions, if θ is zero or 180° (elevation is 90° or −90°) then the azimuth angle is arbitrary; and if r is zero, both azimuth and inclination/elevation are arbitrary.

Also, the azimuth φ is often restricted to the interval (−180°, +180°], or $(-\pi, +\pi]$ in radians, instead of [0, 360°), and the elevation angle is utilized in the inclination plane instead of the inclination/zenith/polar angle.

This is the standard convention for geographic spherical coordinates. In such coordinates, commonly adopted in meteorology, the azimuthal angle (called the *longitude*) is denoted as λ, while the elevation angle (called the *latitude*) is denoted as φ.

5. Conversion of spherical coordinates to Cartesian and cylindrical coordinates

The spherical coordinates (with radius r, inclination θ, azimuth φ) of a point can be obtained from its Cartesian coordinates (x, y, z), see Class 4, as

$$r = \sqrt{x^2 + y^2 + z^2} \ ,$$

$$\varphi = \arctan(y / x) \, ,$$

$$\theta = \arccos(z / r) \, .$$

Function $\varphi = \arctan(y / x)$ must be suitably defined, taking into account the correct quadrant of (x,y), like it was done in the case of polar coordinates, see p. **2**.

Conversely, the Cartesian coordinates x, y, z may be retrieved from the spherical coordinates with (r, θ, φ) as

$$x = r \sin\theta \cos\varphi \, ,$$

$$y = r \sin\theta \sin\varphi \, ,$$

$$z = r \cos\theta \, .$$

Cylindrical coordinates (with radius ρ, azimuth φ, elevation z) may be converted into spherical coordinates (with radius r, inclination θ, azimuth φ) as

$$r = \sqrt{\rho^2 + z^2} \ ,$$

$$\varphi = \varphi \, ,$$

$$\theta = \arctan(\rho / z) \, ,$$

and back as

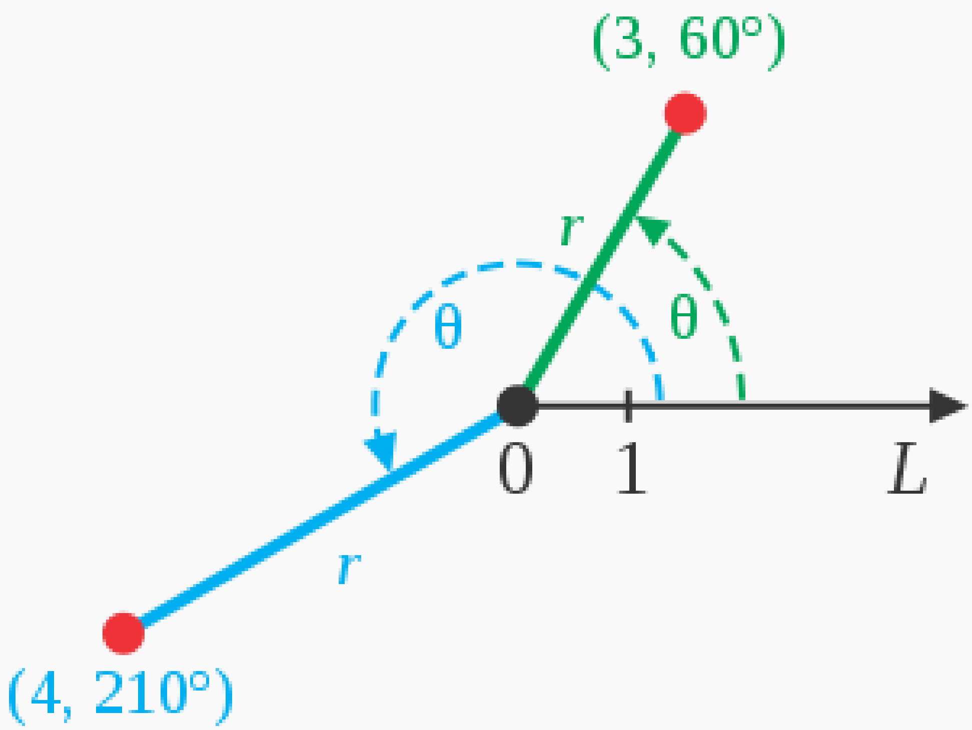
$$\rho = r \sin\theta \, ,$$

$$\varphi = \varphi \, ,$$

$$z = r \cos\theta \, .$$

The above expressions assume that the two systems have the same origin and same reference plane, measure the azimuth angle φ in the same sense from the same axis, and that the spherical angle θ represents the inclination from the cylindrical-coordinate z axis.

Illustration of polar coordinate system



Points in the polar coordinate system with pole 0 and polar axis *L*. Shown are the point with radial coordinate 3 and angular coordinate 60 degrees, or (3,60°) and the point (4,210°).

Relationship between polar and Cartesian coordinates

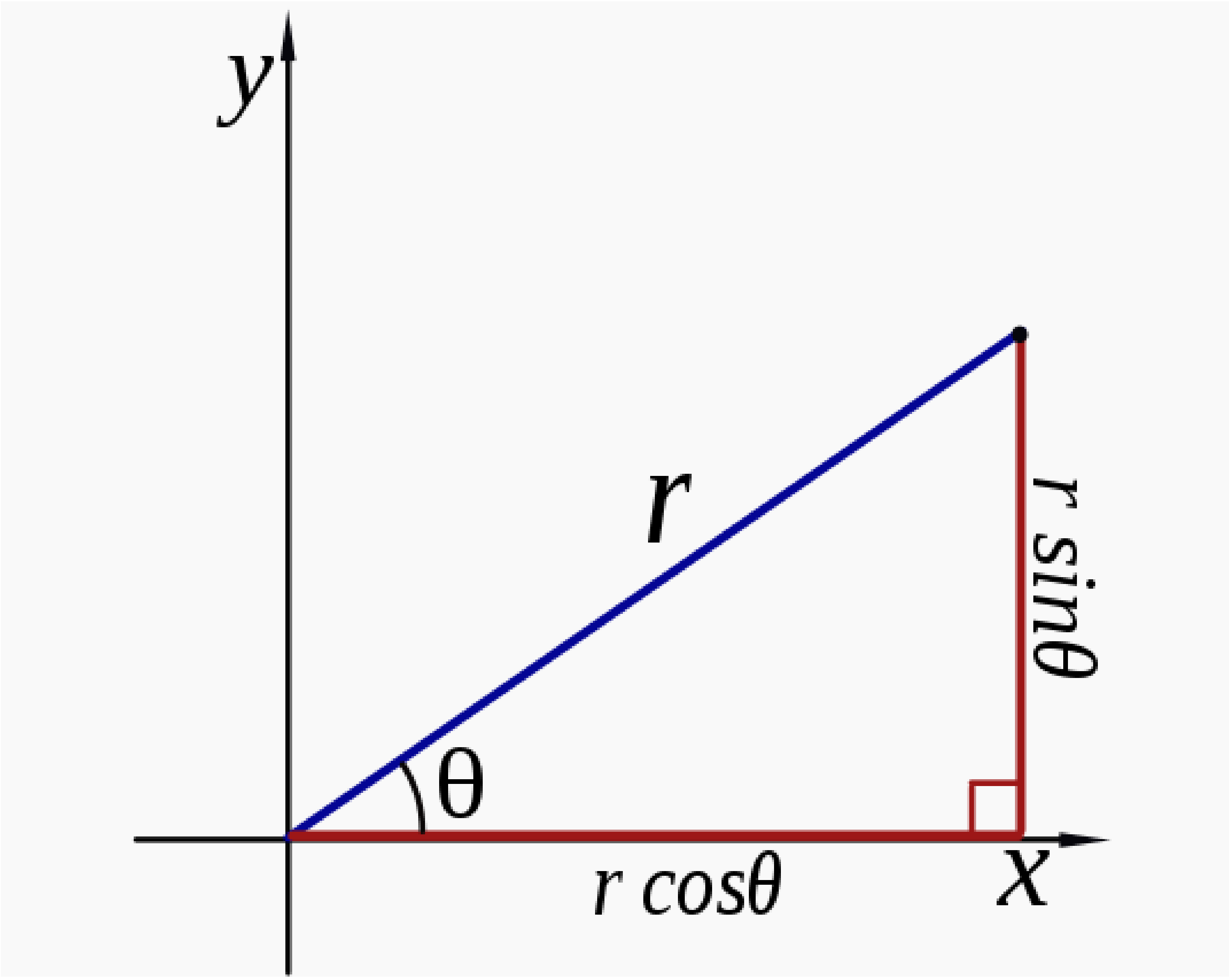
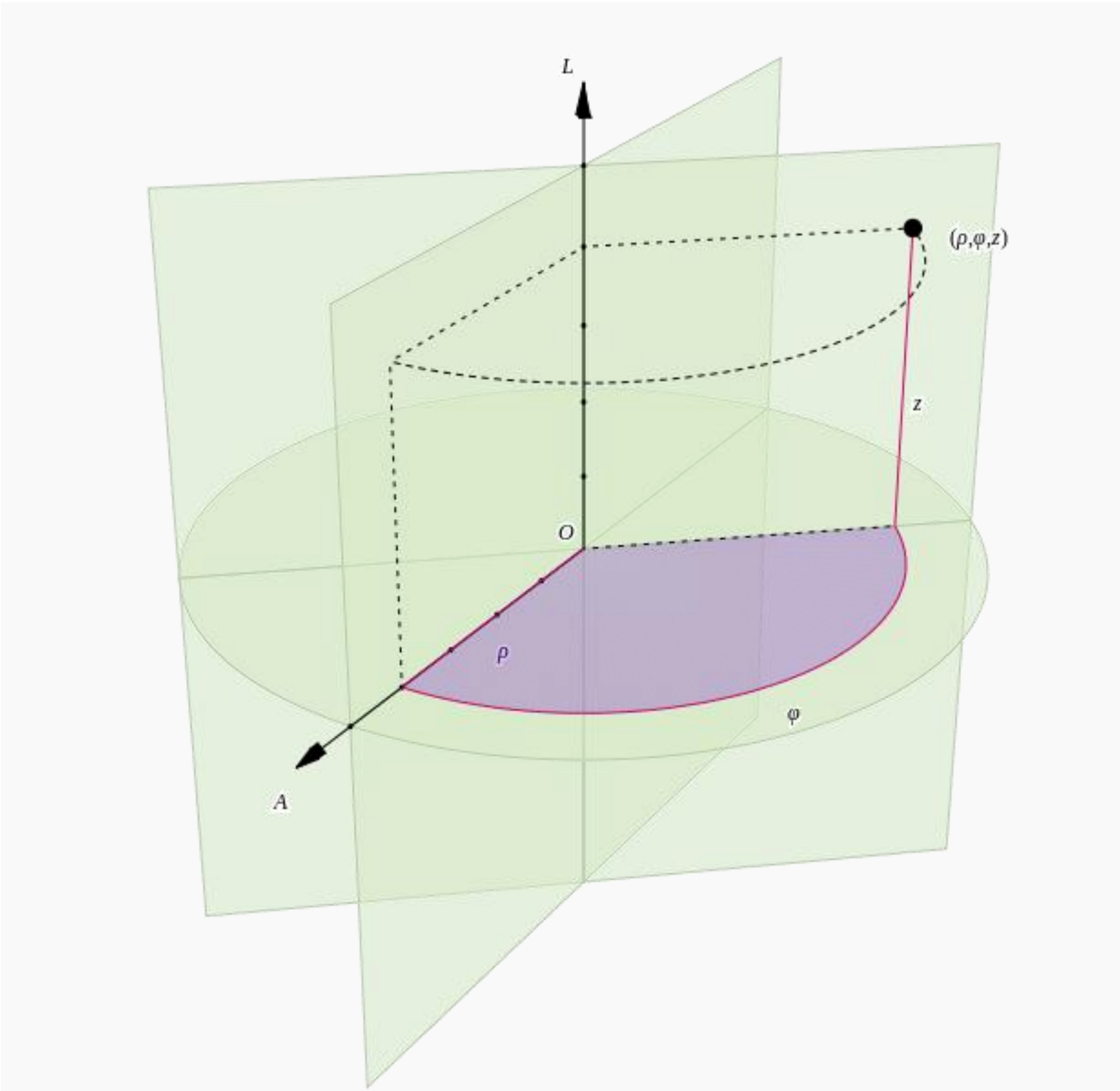
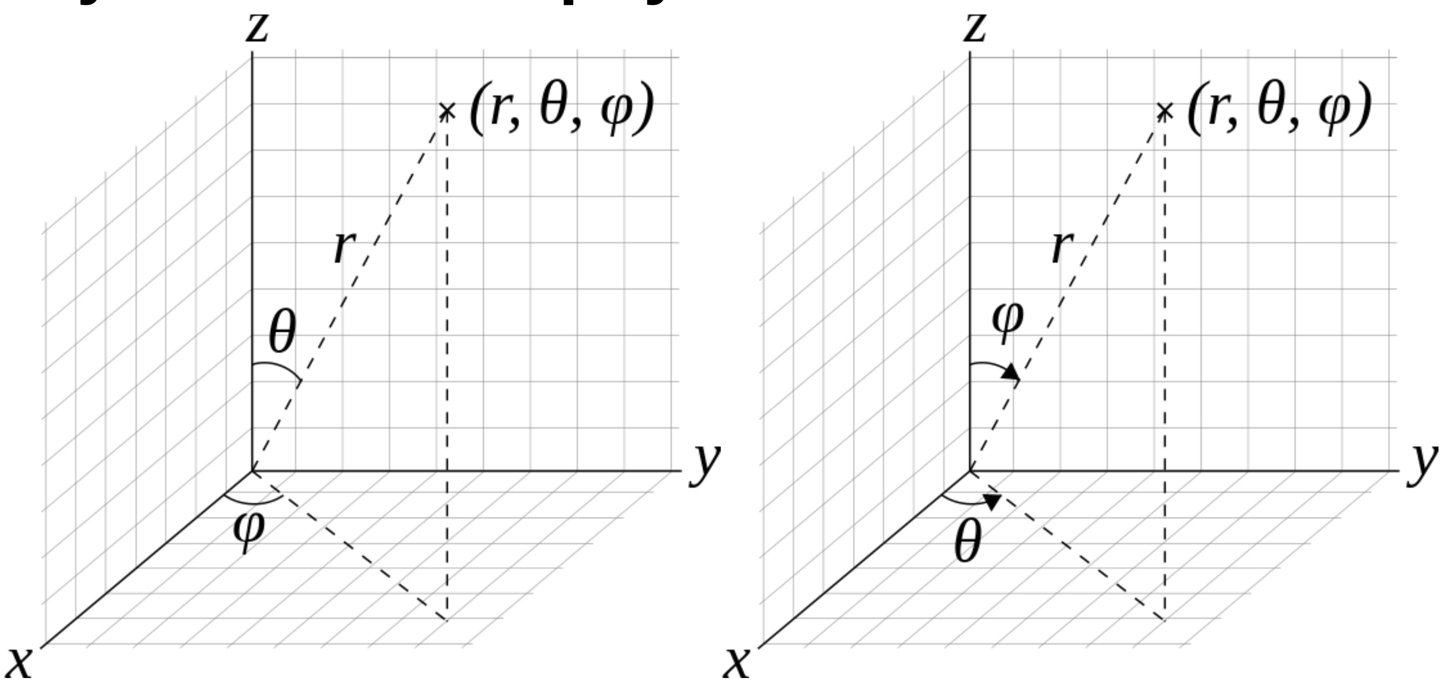


Illustration of cylindrical coordinate system



A cylindrical coordinate system with origin *O*, polar axis *A*, and longitudinal axis *L*. The dot is the point with radial distance $\rho = 4$, angular coordinate $\varphi = 130°$, and height $z = 4$.

Different versions of spherical coordinate system used in physics and mathematics



Note different notations for radial and zenith angles.

From Trapp (2013): Mesoscale Processes in the Atmosphere

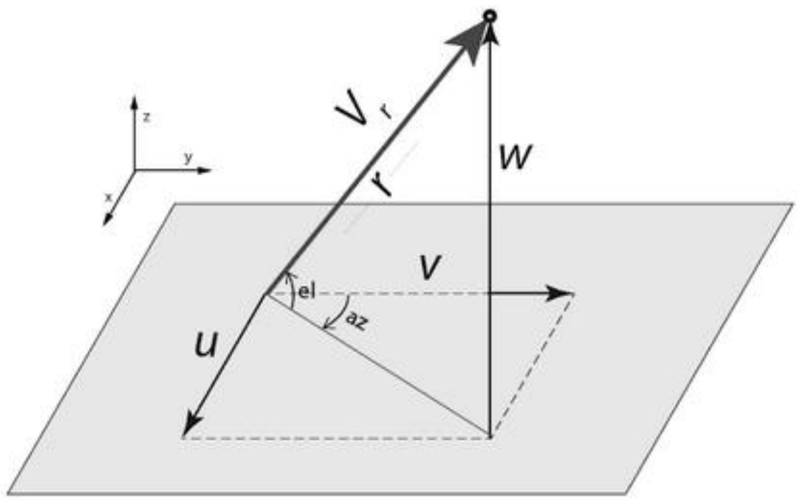


Figure 3.10 Illustration of the geometric relationship between Doppler velocity V_r and Cartesian velocity components u, v, and w. Indicated are the elevation angle (el), azimuth angle (az), and slant range (r).

3.4.1.2 Retrievals from Multiple Doppler Radars

Nearly simultaneous scans by two or more (non-colocated) Doppler radars allow for the retrieval of the complete 3D winds in a spatial domain. The multiple-Doppler wind retrieval technique makes use of spherical geometry to relate the Doppler velocities of the radars to Cartesian wind components: The Doppler velocity measured at some "point" in space by radar *i* is

$$V_{r,i} = u\left(\frac{x - x_i}{r_i}\right) + v\left(\frac{y - y_i}{r_i}\right) + W_p\left(\frac{z - z_i}{r_i}\right), \tag{3.12}$$

where (x, y, z) is the Cartesian location of the point, (x_i, y_i, z_i) is the Cartesian location of the radar, and $r_i = [(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2]^{1/2}$ is the slant range. Equation (3.12) makes use of the coordinate transformations

$$\begin{aligned}
x &= x_i + r_i \sin(az_i)\cos(el_i) \\
y &= y_i + r_i \cos(az_i)\cos(el_i) \\
z &= z_i + r_i \sin(el_i),
\end{aligned} \tag{3.13}$$

Geographical spherical coordinates