Lecture 8. September 9, 2016

Topic: Ekman-model wind-component profiles and corresponding wind hodograph (Ekman spiral) plotted in differently oriented Cartesian coordinate systems. Evaluation of line-parallel and line-normal wind components.

Reading: Section 8.3.4 of Holton and Hakim.

Ekman model solutions

Profiles of the wind components, $u(z)$ and $v(z)$, in the lower atmosphere may be approximately described within the framework of the so-called Ekman model:

$$k\frac{\partial^2 u}{\partial z^2}+f(v-V_g)=0\,,$$

$$k\frac{\partial^2 v}{\partial z^2}-f(u-U_g)=0\,,$$

see Class 7, for notation and additional information.

For the northern hemisphere, in a local Cartesian coordinate system (*X*, *Y*, *Z*), hereafter called conventional, where *X* Cartesian coordinate axis is assumed to be directed eastward and *Y* Cartesian coordinate axis is assumed to be directed northward, these profiles are given by

$$u=U_g-e^{-a_e z}(U_g\cos a_e z+V_g\sin a_e z)\,,$$

$$v=V_g-e^{-a_e z}(V_g\cos a_e z-U_g\sin a_e z)\,,$$

(see Class 7), where *z* is the vertical Cartesian coordinate, $a_e=\sqrt{f/2k}$ is a positive dimensional parameter inversely proportional to the boundary-layer depth scale, u is the wind component along the *X* Cartesian coordinate axis, v is the wind component along the *Y* Cartesian coordinate axis, and U_g, V_g, respectively, are the *x* and *y* components of the geostrophic wind.

According to the Ekman model, *u* tends to U_g and v tends to V_g at large *z*, while both *u* and *v* are zeros at *z*=0.

1. Case of conventional coordinate system

Imagine that you need to calculate and plot Ekman-model wind components $u(z)$ and $v(z)$, and corresponding wind hodograph (which is commonly called the *Ekman spiral*, see Class 7) in the above specified conventional coordinate system for the range of *z* from 0 to 2000 m.

Given are $U_g = 10 \text{ m s}^{-1}$ and $V_g = 5 \text{ m s}^{-1}$ (try to evaluate the resulting geostrophic wind direction in this case), Coriolis parameter $f = 1.25 \cdot 10^{-4} \text{ s}^{-1}$, and eddy diffusivity $k = 10 \text{ m}^2\text{s}^{-1}$. This results in $a_e = 0.0025 \text{ m}^{-1}$, which corresponds to the boundary-layer depth scale of $\pi / a_e \approx 1256 \text{ m}$ (see Class 7). Geostrophic wind components $U_g = 10 \text{ m s}^{-1}$ and $V_g = 5 \text{ m s}^{-1}$ correspond, in conventional terms, to the geostrophic wind with a magnitude (speed) of about 11 m s^{-1} blowing from SWW.

Components of the Ekman-model wind in this case are obtained directly from the solution of the Ekman equations:

$$u = U_g - e^{-a_e z}(U_g \cos a_e z + V_g \sin a_e z)\,,\ v = V_g - e^{-a_e z}(V_g \cos a_e z - U_g \sin a_e z)\,.$$

These components and the wind hodograph in the conventional Cartesian coordinate system with X directed east and Y directed north are shown, respectively, in Figs. ***1i*** and ***1ii*** below.

Figure ***1i***

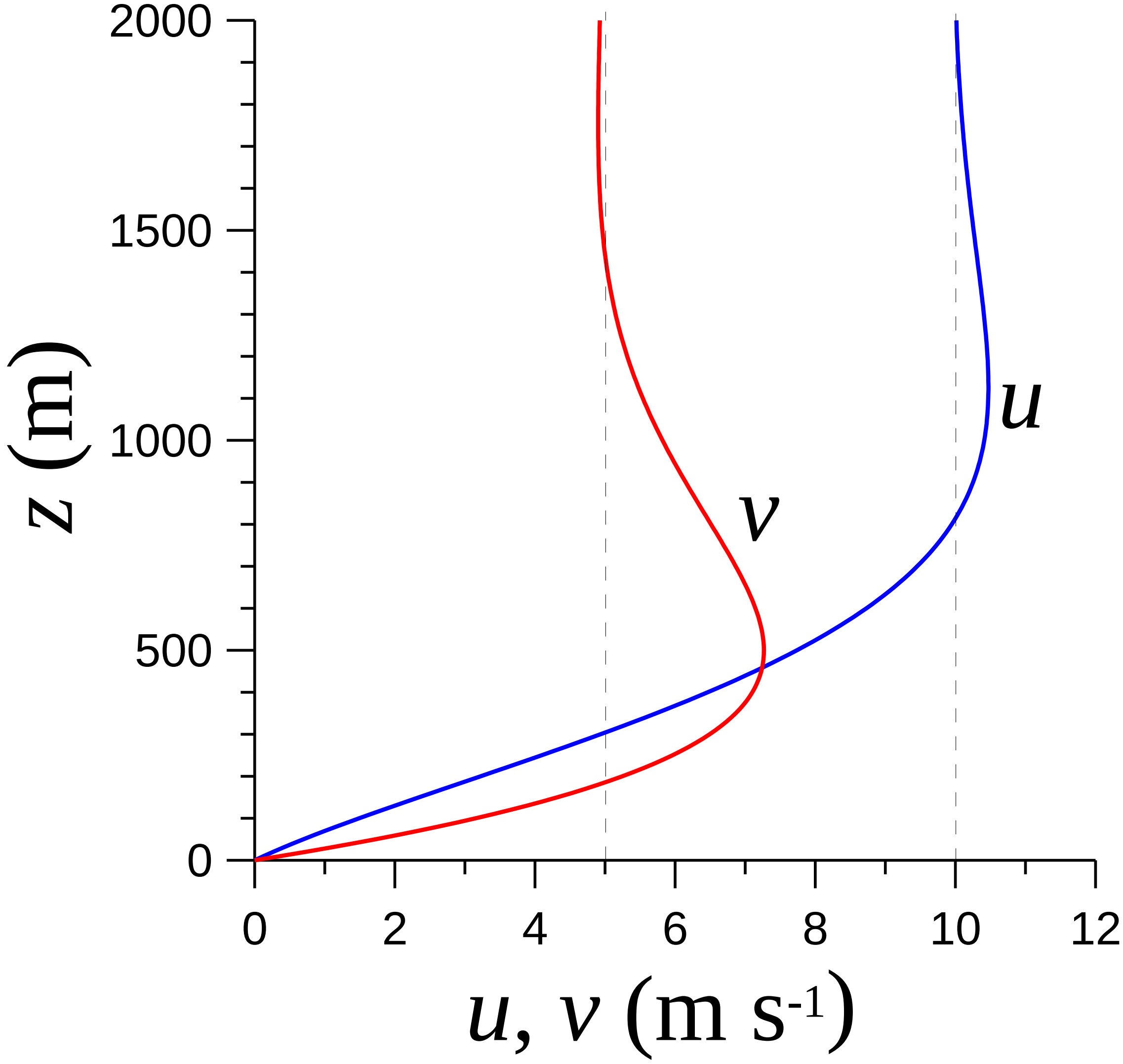
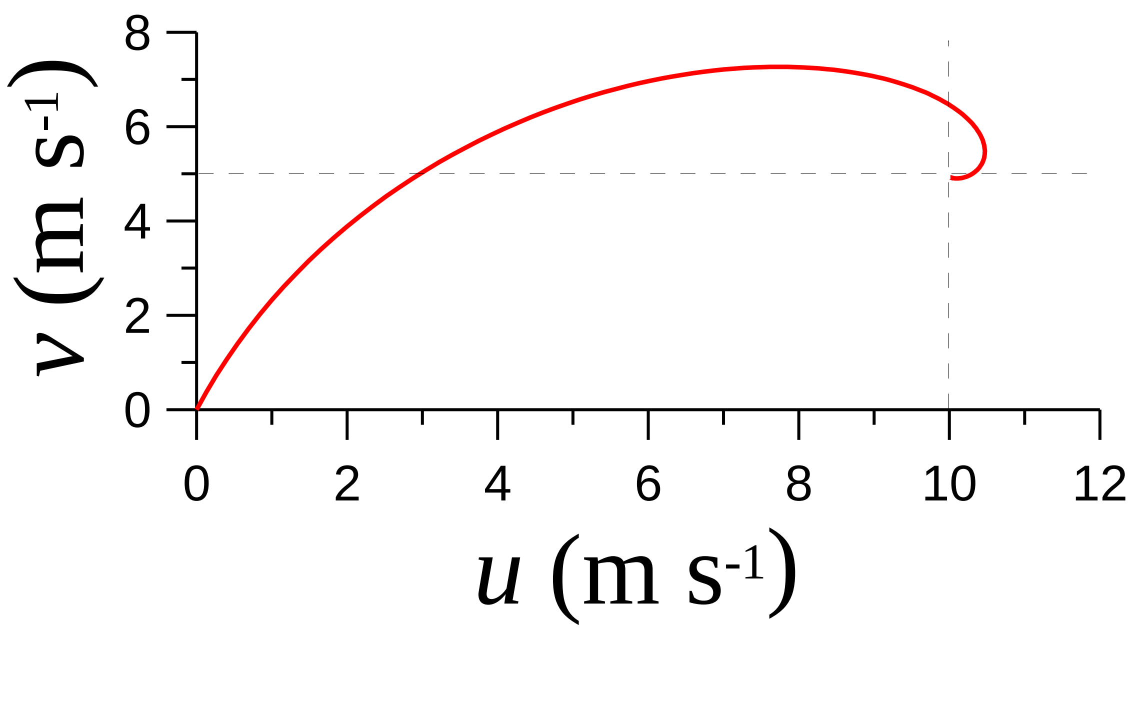


Figure ***1ii***



2. Case of the *X* coordinate axis directed along the geostrophic wind vector

Now do the same wind-profile and hodograph calculation in the new Cartesian system with *X* axis (denote it *X′*) directed along the geostrophic wind (which will have the same magnitude as the geostrophic wind in the previous case), see Class 7). Angle between coordinates (*X′*, *Y′*) and (*X*, *Y*) will be $\alpha = \arctan(V_g / U_g)$ and the geostrophic wind components in the conventional and new systems will be related as (see Class 7):

$$U_g{}' = U_g \cos\alpha + V_g \sin\alpha , \qquad V_g{}' = V_g \cos\alpha - U_g \sin\alpha ,$$

with

$$U_g{}' \equiv G = U_g / \cos\alpha , \qquad V_g{}' = 0 ,$$

and wind components given by (see Figs. ***2i*** and ***2ii*** below):

$$u'(z) = G(1 - e^{-a_e z} \cos a_e z) , \qquad v'(z) = G e^{-a_e z} \sin a_e z .$$

Figure ***2i***

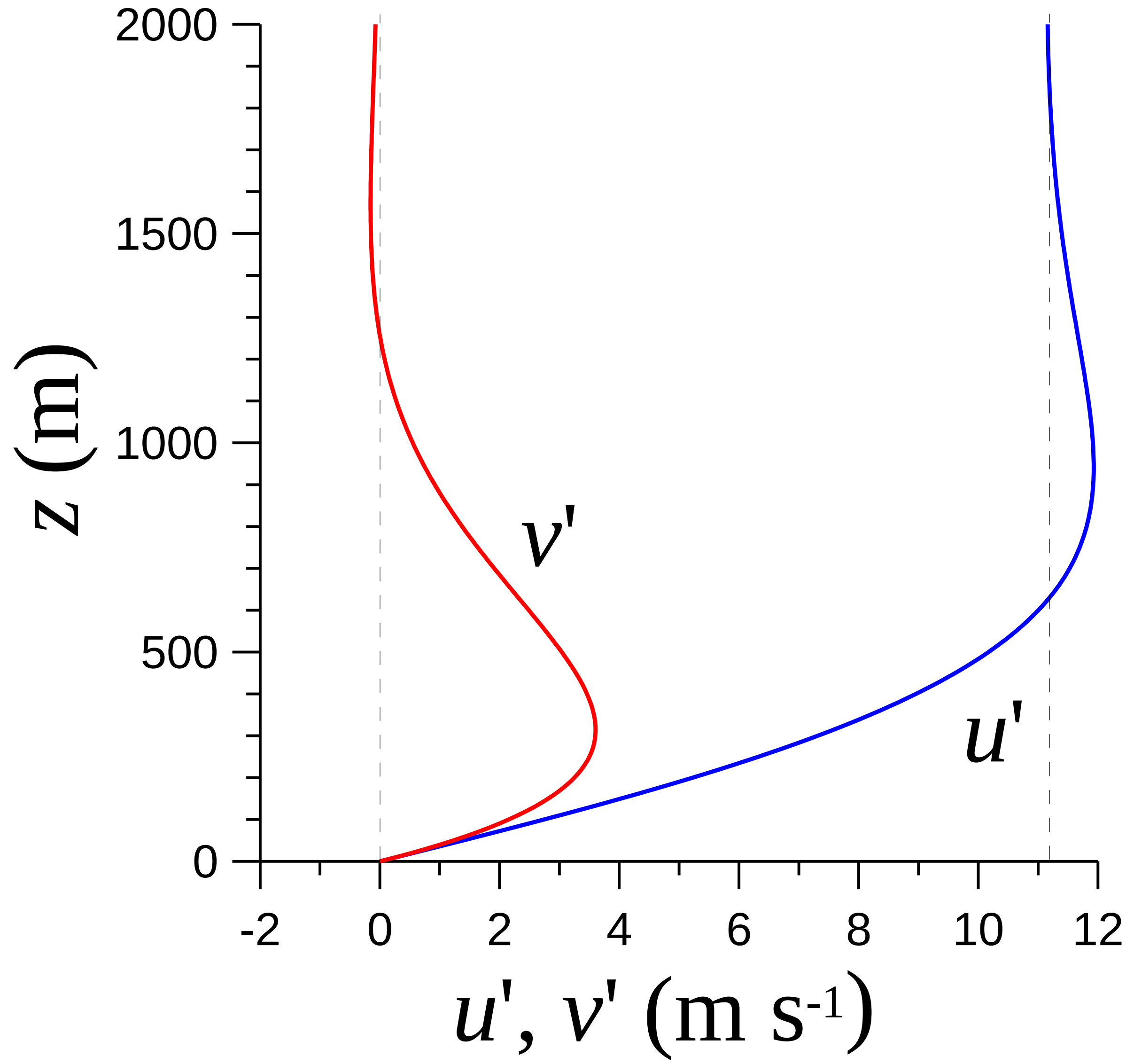
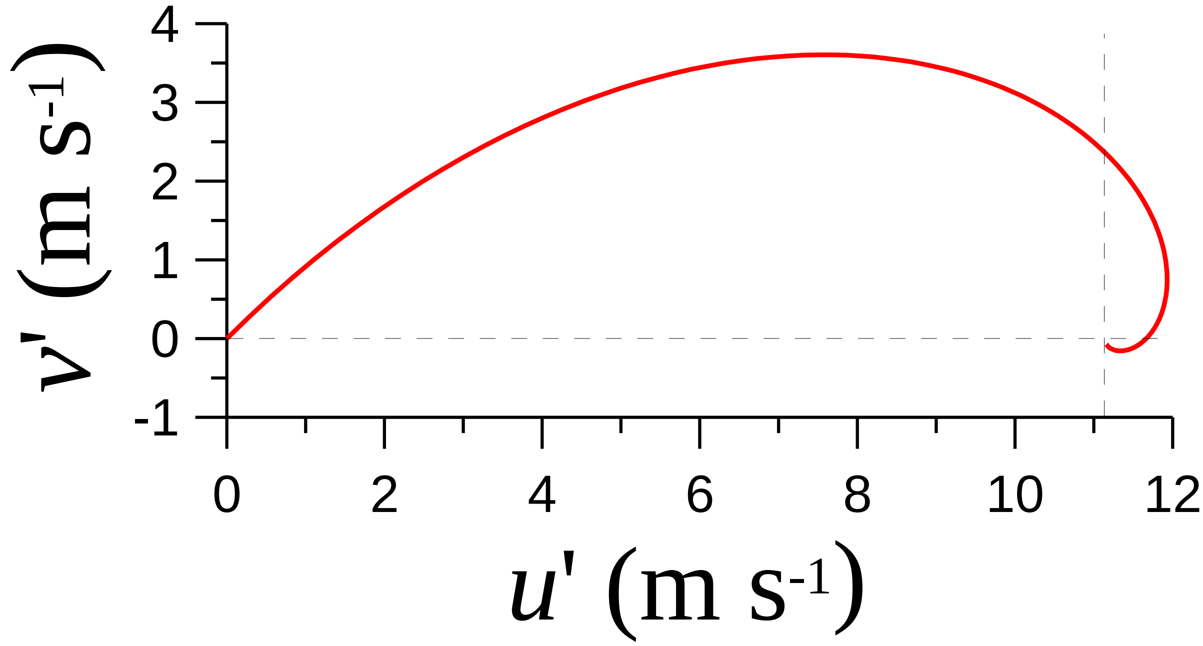


Figure ***2ii***



3. Case of the *X* coordinate axis directed along the surface wind

Consider Ekman-model wind-component profiles and the wind hodograph in yet another coordinate system, with the new Cartesian *X* axis (denote it X'') directed along the near-surface wind and with geostrophic wind having the same magnitude as in the previous two cases.

First we need to evaluate direction of the near-surface wind with respect to the geostrophic wind in the Ekman model formulation corresponding to this case. In the northern hemisphere, as was explained in Class 7, the model near-surface wind is directed at the angle $\pi/4 = 45^{o}$ to the right of geostrophic wind.

This means that in order to direct X'' along the near-surface wind we need to rotate the coordinates used in the case of X' directed along the geostrophic wind by additional $\varphi = \pi/4 = 45^{o}$ counterclockwise.

This provides

$$u'' = u'\cos\varphi + v'\sin\varphi, \qquad v'' = v'\cos\varphi - u'\sin\varphi,$$

and

$$U_g'' = U_g'\cos\varphi + V_g'\sin\varphi, \quad V_g'' = V_g'\cos\varphi - U_g'\sin\varphi.$$

Components of the Ekman-model wind and the wind hodograph calculated using the above expressions are shown, respectively, in Figs. ***3i*** and ***3ii***.

Figure ***3i***

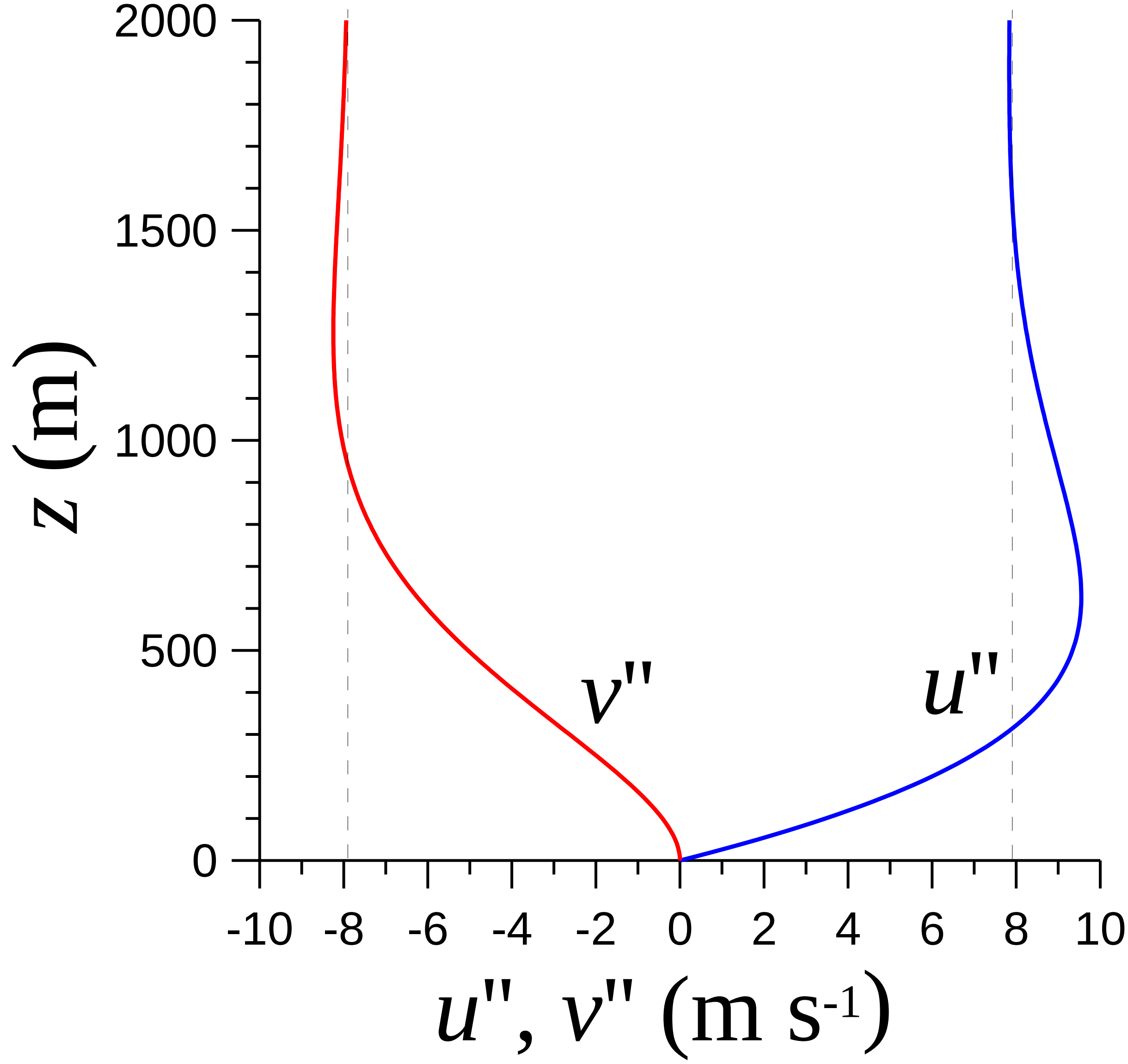
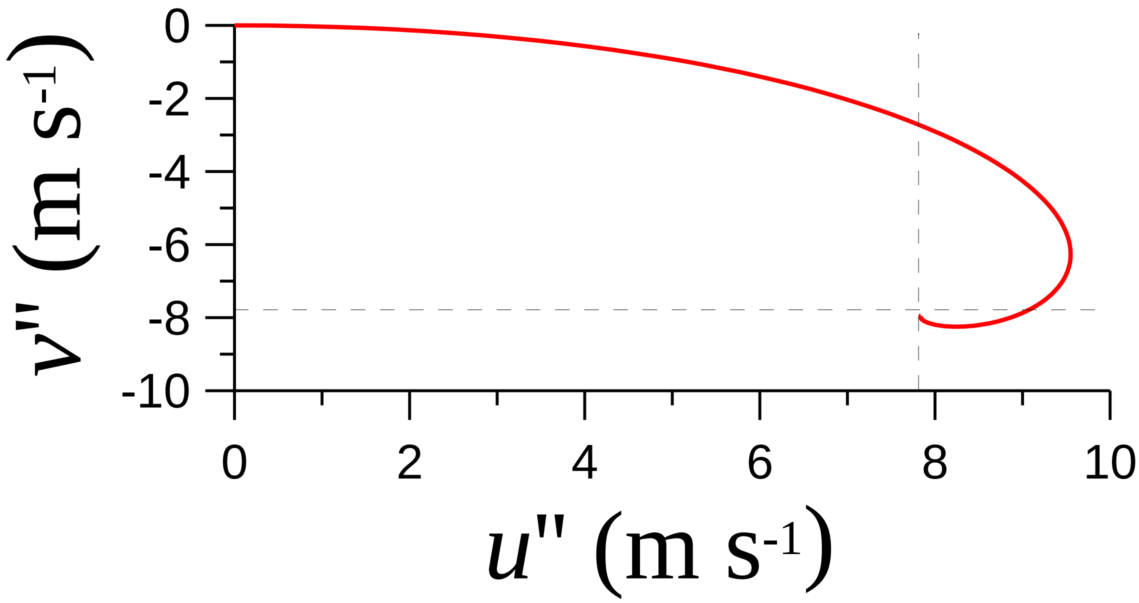


Figure ***3ii***

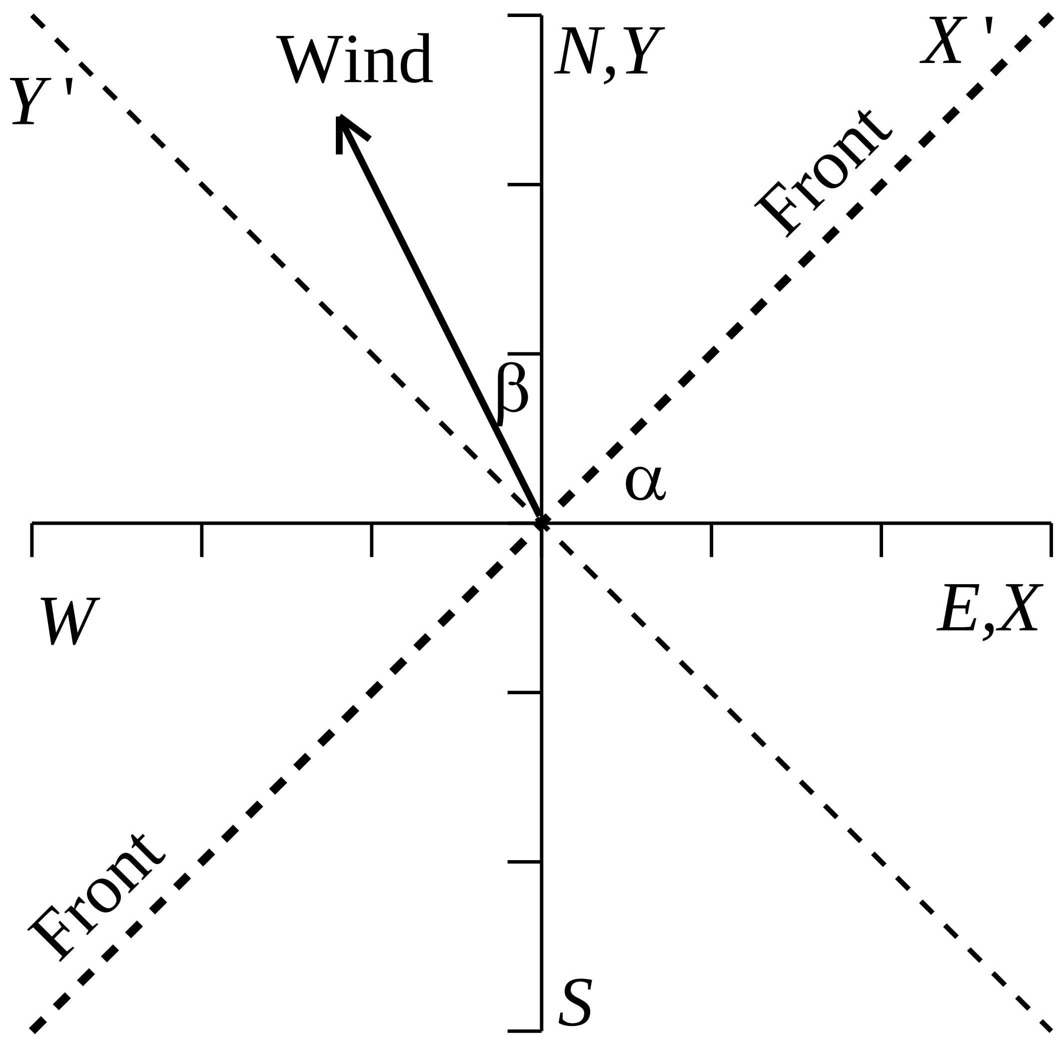


4. Evaluation of line-parallel and line-normal wind components

Atmospheric front is directed from SW to NE, and a 10-m s^{-1} wind is blowing from 150° (approximately SSE). Estimate the front-normal and front-parallel wind components.

Solution

Direction of the wind with respect to the front is illustrated in the plot below. In the (X, Y) coordinate system, the wind has components $u = -V \sin\beta$ and $v = V \cos\beta$, where $V = 10\text{ m s}^{-1}$ is the wind magnitude (speed) and β is the angle between the wind vector and the Y axis (as follows from the information regrading the wind direction in the problem statement, $\beta = 30^{\circ}$, see the plot).



The front-normal and front-parallel wind components are the components of the wind vector in the rotated (X', Y') system aligned with the front, in which the X' axis is directed along the front and Y' axis normal to the front according to the right-hand convention. The front-normal component is along Y', and front-parallel component is along X'. Given the direction of the front, the angle α between the X axis and the front (X' axis) is 45°. Let us denote wind vector components in the (X', Y') system as u' (this will be the front-parallel component) and v'

(this will be the front-normal component). According to the rules of the vector-component transformation in the rotated coordinate system, wind components u, v and u', v' are related as

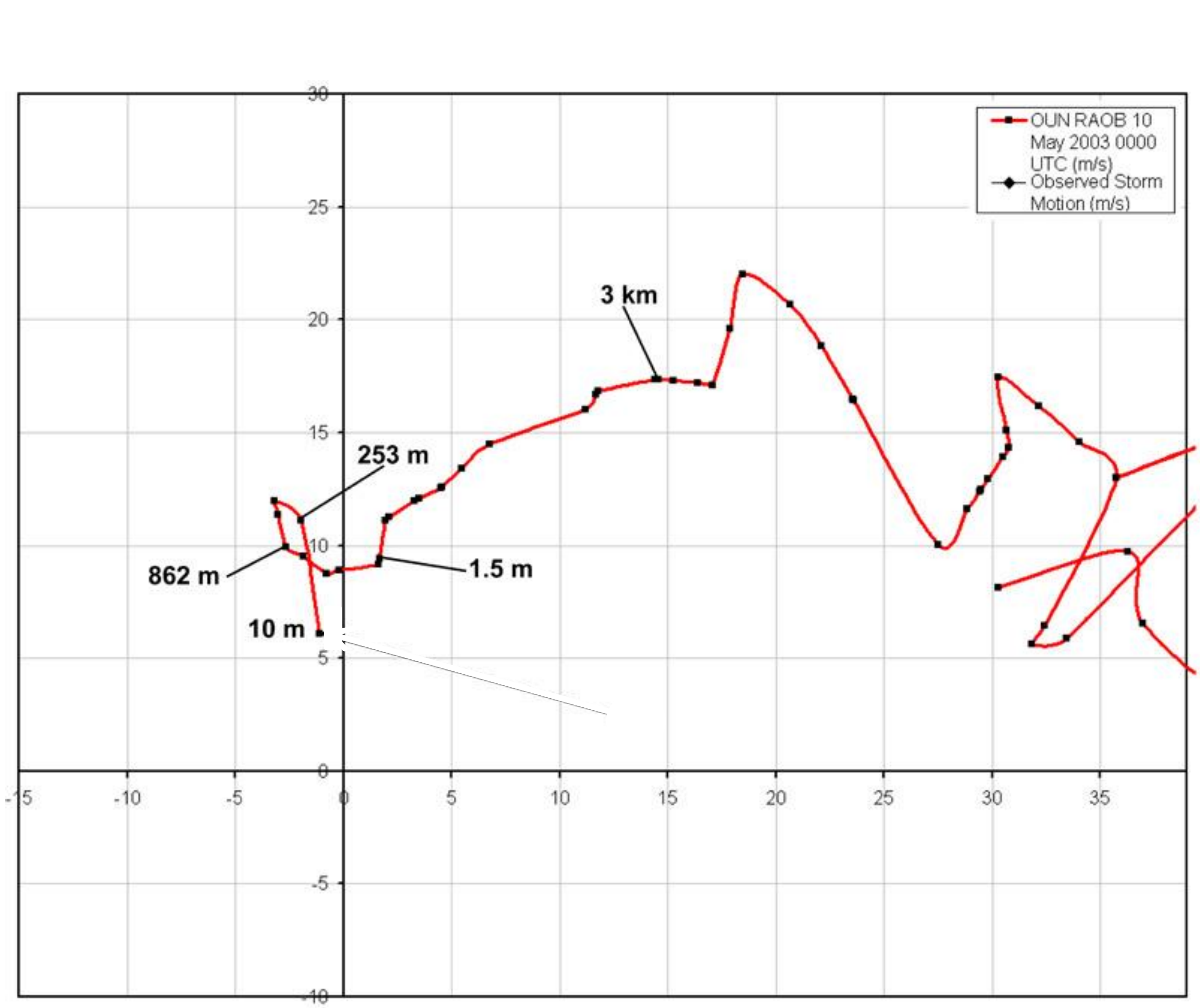
$$u' = u\cos\alpha + v\sin\alpha ,$$

$$v' = -u\sin\alpha + v\cos\alpha ,$$

which provides the following front-parallel and front-normal wind velocity components:

$$u' = -V\sin\beta\cos\alpha + V\cos\beta\sin\alpha = -10\cdot\frac{1}{2}\cdot\frac{\sqrt{2}}{2} + 10\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = 2.5\sqrt{2}(\sqrt{3}-1) \simeq 2.6 \text{ m s}^{-1} ,$$

$$v' = V\sin\beta\sin\alpha + V\cos\beta\cos\alpha = 10\cdot\frac{1}{2}\cdot\frac{\sqrt{2}}{2} + 10\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = 2.5\sqrt{2}(\sqrt{3}+1) \simeq 9.7 \text{ m s}^{-1} .$$



OUN RAOB 10 May 2003 0000 UTC (m/s)
Observed Storm Motion (m/s)
3 km
253 m
862 m
1.5 m
10 m
30
25
20
15
10
5
0
-5
-10
-15
-10
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0
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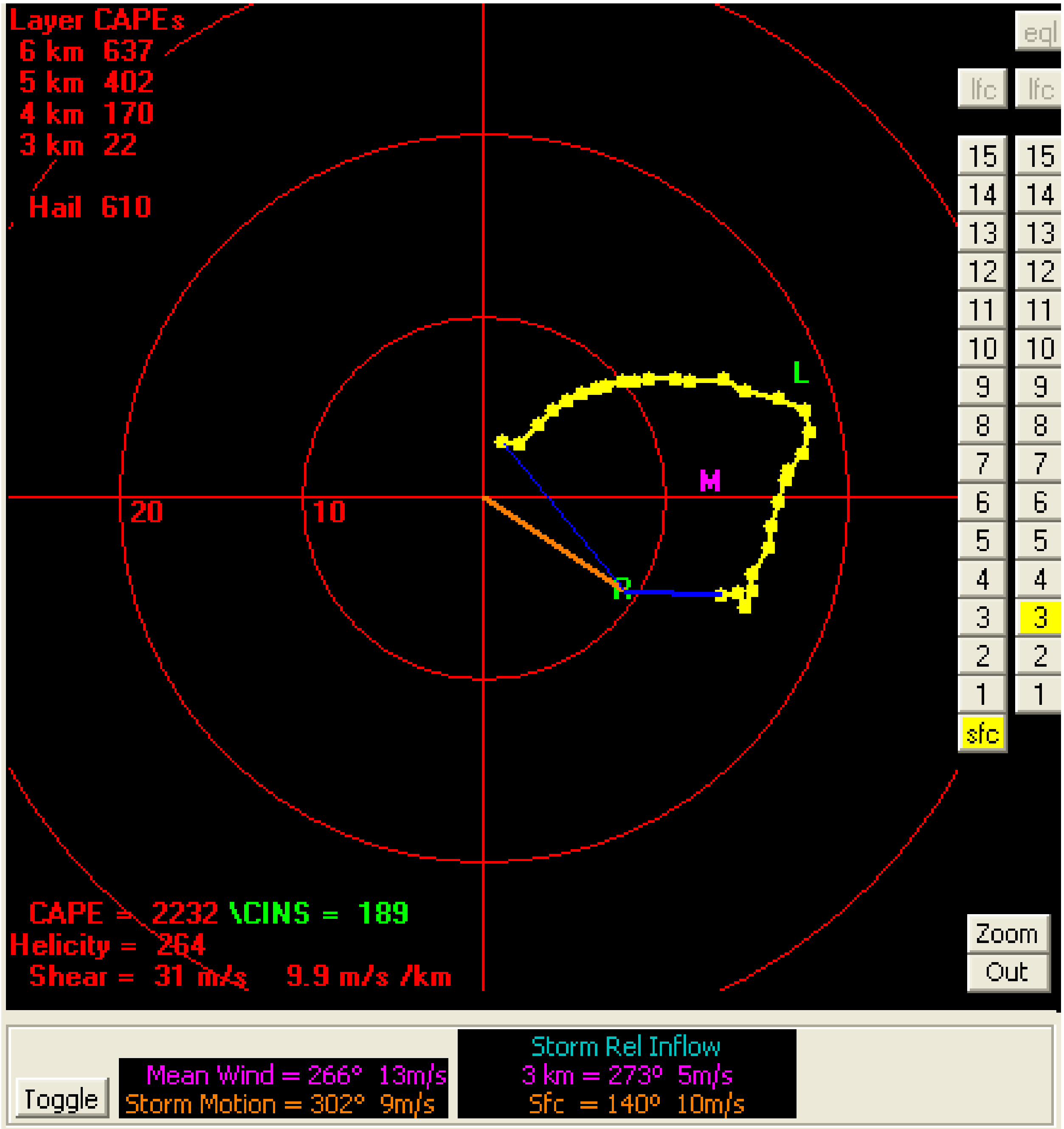
30
35


Layer CAPEs
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5 km 402
4 km 170
3 km 22
Hail 610
20
10
L
M
eql
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lfc
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
sfc
CAPE = 2232 \CINS = 189
Helicity = 264
Shear = 31 m/s 9.9 m/s /km
Zoom
Out
Toggle
Mean Wind = 266° 13m/s
Storm Motion = 302° 9m/s
Storm Rel Inflow
3 km = 273° 5m/s
Sfc = 140° 10m/s