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METR 2103, Physical Mechanics
Fall 1996

Solution of Second-Order, Homogeneous Linear Ordinary Differential Equations with Constant Coefficients

Consider the following second-order, homogeneous, linear, ordinary differential equation with constant coefficients:

$$y'' + ay' + by = 0 \tag{1}$$

where a and b are constants and a prime indicates a total derivative with respect to time (it could be with respect to x or z as well). The associated characteristic equation is given by

$$\lambda^2 + a\lambda + b = 0 \tag{2}$$

where λ is the so-called Eigenvalue, or characteristic mode of the system (in the case of an oscillation, λ is the fundamental frequency of vibration). The solution to (1) depends upon the roots of the quadratic equation (2)

$$\lambda = -\frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 - 4b}\ . \tag{3}$$

We must examine all three possibilities, and for simplicity, let λ_1 *and* λ_2 be the two roots of (3).

Case I: Discriminant in (3) Equal to Zero ($a^2 - 4b = 0$)

In this case, there is only one value of λ and the solution to (1) is written as

$$y(t) = (A + Bt)e^{\lambda t}$$

where A and B are constants determined by the initial or boundary conditions.

Case II: Discriminant in (3) is Negative ($a^2 - 4b < 0$)

In this case, the roots to (3) are complex. Letting -p = a/2 and q = $\frac{1}{2}\sqrt{a^2 - 4b}$, the roots to (3) can be written more compactly as $\lambda_1 = p + iq$ *and* $\lambda_2 = p - iq$ and the general solution as

$$y(t) = e^{pt}\left[A\cos(qt) + B\sin(qt)\right]\ .$$

Case III: Discriminant in (3) is Positive ($a^2 - 4b > 0$)

In this case, the roots to (3) are real - let's call them λ_1 *and* λ_2 . The solution can be written as

$$y(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}\ .$$