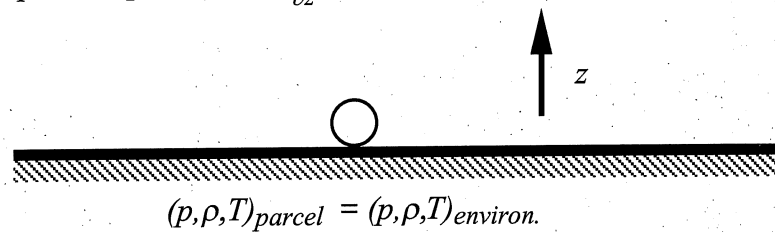


METR 4433, Mesoscale Meteorology Spring 1997

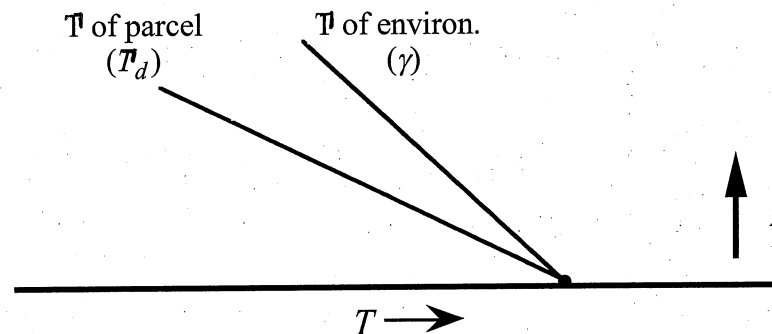
Review of Buoyancy / Stability Theory

A. The theory behind gravity waves can be traced to the "parcel method" for evaluating atmospheric stability. Recall that the parcel method is a way to assess the stability of air that is displaced vertically, and it doesn't require mathematics. In it, we simply compare the atmospheric lapse rate with that of the rising parcel.

$$\text{Let } \gamma = \text{atmospheric lapse rate} = -\frac{\partial T}{\partial z}$$



There is no net force on the parcel since it has the same T , p , and ρ as the environment. If the parcel is "kicked" upward, it will expand and cool at the dry adiabatic lapse rate IF it remains unsaturated. If the parcel cools at a rate faster than the environment, i.e., if $\mathcal{T}_d > \gamma$ (\mathcal{T}_d = lapse rate of parcel), then at the stopping point, it will be colder or less buoyant than its surroundings (assuming that the pressures come into equilibrium quickly)... the parcel will sink.



This is, of course, a stable situation. When the parcel reaches its original level, it overshoots it due to its inertia and sinks a little further...then, it'll be warmer and thus rise. This up/down oscillating motion about the level of neutral buoyancy continues until viscosity robs the motion of its energy.

In the unstable case, the parcel simply keeps going (accelerated by buoyancy forces--the difference in density between parcel and atmosphere \times gravity). In the neutral case, the parcel is always in equilibrium. **Note:** Oscillations only occur in the stable case!

$$\gamma < \Gamma_d \rightarrow \text{stable} \quad \gamma > \Gamma_d \rightarrow \text{unstable} \quad \gamma = \Gamma_d \rightarrow \text{neutral}$$

B. Let's now quantify the stable case mathematically by looking at equations for the parcel and the environment:

Environment: Assumed to be in hydrostatic balance...no compensating motions occur as the parcel moves.

$$\frac{\partial \bar{p}}{\partial z} = -\bar{\rho}g \quad (\bar{ }) = \text{environment, which is a function of height only.}$$

Parcel: The parcel will undergo an acceleration by virtue of the PGF force and gravity acting on it:

$$\frac{dw^*}{dt} = \frac{-1}{\rho^*} \frac{\partial p^*}{\partial z} - g \quad ()^* = \text{parcel}$$

Now, we assume that the parcel and environment have the same pressure (rapid adjustment), so that

$$\frac{\partial \bar{p}}{\partial z} = \frac{\partial p^*}{\partial z}$$

Substituting these identities into the parcel equation gives

$$\frac{dw^*}{dt} = \frac{-1}{\rho^*} (-\bar{\rho}g) - g$$

or

$$\frac{dw^*}{dt} = g \left(\frac{\bar{\rho} - \rho^*}{\rho^*} \right) \quad (\rho^* = \text{total density of the parcel})$$

This equation can be written in a more familiar form if we express the parcel density ρ^* as the sum of a base state $\bar{\rho}(z)$ and a deviation ρ' . Then,

$$(\rho^* = \bar{\rho} + \rho')$$

$$\frac{dw^*}{dt} = g \left[\frac{\bar{\rho} - \bar{\rho} - \rho'}{\bar{\rho} + \rho'} \right] = -g \frac{\rho'}{\bar{\rho} + \rho'} \approx -g \frac{\rho'}{\bar{\rho}}$$

Recall that $\frac{\rho'}{\bar{\rho}} = -\frac{\theta'}{\bar{\theta}}$. Thus,

$$\frac{dw}{dt} \cong g \frac{\theta'}{\bar{\theta}} \quad (\text{drop } * \text{ on } w)$$

Goal: Express buoyancy in terms of stability... $d\bar{\theta}/dz$.

Let's write $\bar{\theta}(z)$ in terms of a Taylor series expansion:

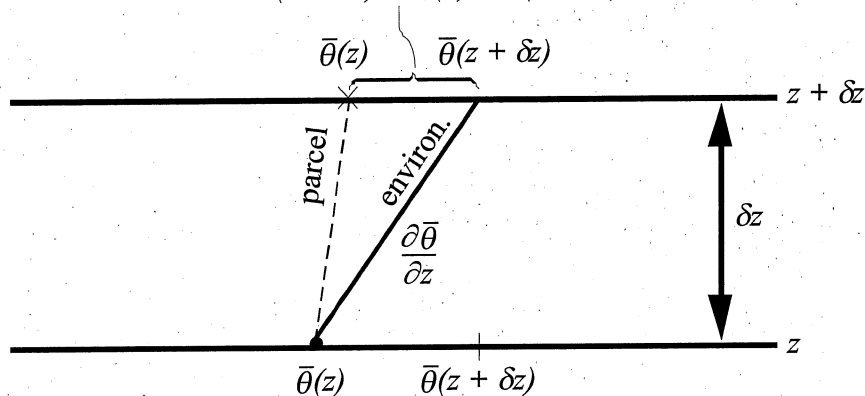
$$\bar{\theta}(z + \delta z) = \bar{\theta}(z) + \frac{\partial \bar{\theta}}{\partial z} \delta z + \frac{\partial^2 \bar{\theta}}{\partial z^2} \frac{(\delta z)^2}{2!} + \dots$$

If δz is small, we can neglect the higher-order terms to yield,

$$\bar{\theta}(z + \delta z) \cong \bar{\theta}(z) + \frac{\partial \bar{\theta}}{\partial z} \delta z$$

Now, if the parcel moves adiabatically, it conserves its potential temperature (carries its original potential temperature to the new level). Thus, if it starts at the potential temperature of the environment, we have

$$\theta'(z + \delta z) = \bar{\theta}(z) - \bar{\theta}(z + \delta z)$$



Thus,

$$\theta'(z + \delta z) = -\frac{\partial \bar{\theta}}{\partial z} \delta z$$

and our buoyancy term becomes

$$\frac{g}{\bar{\theta}} \left(-\frac{\partial \bar{\theta}}{\partial z} \delta z \right) \equiv -B^2 \delta z$$

where we have defined

$$B^2 \equiv \frac{g}{\bar{\theta}} \frac{d\bar{\theta}}{dz} = g \frac{d \ln \bar{\theta}}{dz}$$

which is a measure of static stability---called the Buoyancy Frequency. (Note the units, s⁻²).

Now, we can write our equation as

$$\frac{dw}{dt} = -B^2 \delta z$$

But, $w = \frac{d}{dt}(\delta z)$, so we have

$$\frac{d^2}{dt^2}(\delta z) = -B^2 \delta z$$

This is a simple ODE that can be integrated.

If $B^2 > 0$, ($\frac{d\bar{\theta}}{dz} > 0 \Rightarrow$ stable atmosphere) the solution is of the form

$$\delta z = c_1 \cos(Bt) + c_2 \sin(Bt)$$

\Rightarrow parcel displacement is oscillatory about its initial level with period =

$$\frac{2\pi}{B} \sim \frac{2\pi}{1.2 \times 10^{-2} \text{ s}^{-1}} \sim \underline{8 \text{ min.}}$$

If $B^2 = 0$, there is no acceleration (neutral).

If $B^2 < 0$, the solution is of the form

$$\delta z = c_1 e^{|B|t}$$

\Rightarrow exponential increase in parcel height with time \Rightarrow unstable situation.

Moral: The behavior of a displaced parcel is governed by the stability, or more accurately by

$B^2 = \frac{g}{\bar{\theta}} \frac{d\bar{\theta}}{dz}$. We'll see that this result is the basis for internal gravity waves.